King Saud University
Department of Mathematics
M-203[Final Examination]
(Differential and Integral Calculus)
(First-Semester 1428/29)

Max.Marks:50 Time:3hrs

Marking Scheme: Q.No:1[3+4+5+4], Q.No2:[4+4+4+4], Q.No:3[4+4+4+6]

Q.No: 1 (a) Discuss the convergence of the sequence $\{\ln(2n)-\ln(2n-1)\}$.

Solution: $$\lim_{n\to\infty}[\ln(2n)-\ln(2n-1)] = \lim_{n\to\infty}\left[\ln\left(\frac{2n}{2n-1}\right)\right] = \lim_{n\to\infty}(\ln(1)) = 0.$$

(b) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{n}{2n^2-1}$ is absolutely convergent, Conditional ly convergent, or divergent.

Solution: Since $\frac{n}{2n^2-1} > \frac{1}{2n}$ and $\sum\frac{1}{2n}$ is divergent

$\Rightarrow \sum_{n=1}^{\infty}\left|(-1)^n\frac{n}{2n^2-1}\right|$ is divergent.

Let us now try Alternating series test:

(i) $\lim_{n\to\infty}\frac{n}{2n^2-1}=0$,

(ii) $f(x)=\frac{x}{2x^2-1}$, $f'(x)<0$ for $x>1$.

Hence $\sum_{n=1}^{\infty}(-1)^n\frac{n}{2n^2-1}$ is conditionally convergent.

(c) Find the interval of convergence and radius of convergence of the power Series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n3^n}(x-1)^n$.

Solution: $$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(n+1)3^{n+1}}\times\frac{n3^n}{(x-1)^n}\right| = \lim_{n\to\infty}\frac{1}{3}|x-1| = \frac{1}{3}|x-1|$$

For convergence $\frac{1}{3}|x-1|<1 \Rightarrow |x-1|<3$

$\Rightarrow -3<x-1<3$

$\Rightarrow -2<x<4$.

Check convergence at $x=-2$,

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n3^n}(-2-1)^n = \sum_{n=1}^{\infty}(-1)^{2n}\frac{1}{n3^n}(3)^n = \sum_{n=1}^{\infty}\frac{1}{n} \quad \textbf{(Divergent)}$$

Check convergence at $x=4$

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n3^n}(4-1)^n = \sum(-1)^n\frac{3^n}{n3^n} = \sum(-1)^n\frac{1}{n} \quad \textbf{(Convergent)}$$

Therefore Interval of convergence is $(-2,4]$. Radius of Convergence =3.

(d) Find Maclaurin's series for the function $f(x)=\frac{1}{(1+x)^2}$.

Solution: $f(x)=\frac{1}{(1+x)^2} \Rightarrow f(0)=1$

$$f(x)=(1+x)^{-2} \Rightarrow f'(x)=(-2)(1+x)^{-3}(1) \Rightarrow f'(0)=-2$$
$$f''(x)=(-2)(-3)(1+x)^{-4} \Rightarrow f''(0)=6$$
$$f'''(x)=(-2)(-3)(-4)(1+x)^{-5} \Rightarrow f'''(0)=-24$$
$$f^{iv}(x)=(-2)(-3)(-4)(-5)(1+x)^{-6} \Rightarrow f'''(0)=120$$

and so on............

Therefore $f(x)=1-2x+\frac{6}{2!}x^2-\frac{24}{3!}x^3+\frac{120}{4!}x^5$ ------------

Q.No: 2 (a) Sketch the region of integration and evaluate the integral $\int_0^{\sqrt{\pi/2}}\int_x^{\sqrt{\pi/2}} Cos(y^2)dydx$.

Solution: $$\int_0^{\sqrt{\pi/2}}\int_x^{\sqrt{\pi/2}} Cos(y^2)dydx=\int_0^{\sqrt{\pi/2}}\int_0^{y} Cos(y^2)dxdy=\int_0^{\sqrt{\pi/2}} yCos(y^2)dy$$
$$=\frac{1}{2}[Sin\left(\frac{\pi}{2}\right)-Sin(0)]=\frac{1}{2}.$$

(b) Find the area of the surface of the portion of paraboloid $z=x^2+y^2$ that Is cut off by the plane $z=1$.

Solution: Surface Area= $\iint_R \sqrt{1+(f_x)^2+(f_y)^2}\,dA.$

Here $z=x^2+y^2=f(x,y) \Rightarrow f_x(x,y)=2x\ \&\ f_y(x,y)=2y$.

$$=\iint_R \sqrt{1+(2x)^2+(2y)^2}\,dA=\iint_R \sqrt{1+4(x^2+y^2)}\,dA.$$
$$=\int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\,rdrd\theta=2\pi\left[\frac{1}{12}\left(1+4r^2\right)\right]_0^1$$
$$=\frac{\pi}{6}\left(5\sqrt{5}-1\right)$$

(c) A lamina has the shape of the region bounded by $y=\sqrt{a^2-x^2}$ and the density is directly proportional to the distance from x-axis. Find the mass of the lamina.

Solution: Mass of a lamina= $\iint_R \delta(x,y)dA=\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} kydydx=\frac{k}{2}\int_{-a}^{a}\left(a^2-x^2\right)dx$

$$=2\left[\frac{k}{2}\int_0^a\left(a^2-x^2\right)dx\right]=k\left[a^2x-\frac{x^3}{3}\right]_0^a$$
$$=\frac{2}{3}ka^3.$$

(d) Evaluate the integral $\int_0^4\int_0^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dzdydx$.

Solution: **<u>On converting this integral into spherical we get</u>**:

$$= \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{32}} \rho\rho^2 Sin\varphi d\rho d\varphi d\theta == \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{32}} \rho\rho^2 Sin\varphi d\rho d\varphi d\theta = \int_0^{\pi/2}\int_0^{\pi/4}\frac{1}{4}\left(4\sqrt{2}\right)^4 Sin\varphi d\varphi d\theta$$

$$= 256\int_0^{\pi/2}\left[-Cos\varphi\right]_0^{\pi/4} d\theta = 256\left(-\frac{1}{\sqrt{2}}+1\right)\int_0^{\pi/2} d\theta = 37.49\pi$$

<u>On converting the given integral into Cylindrical we get:</u>

$$\int_0^4\int_0^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dzdydx = \int_0^{\pi/2}\int_0^4\int_r^{\sqrt{32-r_2}} \sqrt{r^2+z^2}\,rdzdrd\theta$$

Q.No: 3 (a) Show that the integral $\int_C ye^{xy}dx + (xe^{xy} - 2y)dy$ is independent of path by finding the potential function.

Solution: We need to show $\vec{F}(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}$

$$\Rightarrow \vec{F}(x,y) = ye^{xy}\,\vec{i} + (xe^{xy} - 2y)\,\vec{j} = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}$$

$$\Rightarrow f_x(x,y) = ye^{xy} \quad \text{..........................(1)}$$

$$\Rightarrow f_y(x,y) = xe^{xy} - 2y \text{........................(2)}$$

Integrating (1) with respect to x and (2) with respect to y, we get

$$f(x,y) = ye^{xy}\frac{1}{y} + g(y)\text{.....................(3)}$$

$$f(x,y) = xe^{xy}\frac{1}{x} - y^2 + h(y)\text{...............(4)}$$

Comparing (3) and (4), we get

$$g(y) = -y^2 + h(y)$$

Therefore $f\ (x,y) = e^{xy} - y^2 + c$ which is a potential function.

(b) Evaluate the integral $\oint_C y^2dx + 3xydy$, where C is the boundary of the region that lies inside the circle $x^2 + y^2 = 4$ and outside the circle $x^2 + y^2 = 1$ in upper half plane.

Solution: We apply Green's theorem and we have

$$\oint_C y^2dx + 3xydy = \iint_R \left[\frac{\partial(3xy)}{dx} - \frac{\partial(y^2)}{dy}\right]dA = \iint_R (3y-2y)dA = \iint_R ydA$$

$$= \int_0^{\pi}\int_1^2 rSin(\theta)rdrd\theta = \int_0^{\pi}\left[\frac{r^3}{3}\right]_1^2 Sin(\theta)d\theta = \frac{7}{3}\int_0^{\pi} Sin(\theta)d\theta = \frac{14}{3}$$

(c) Use Stokes's theorem to evaluate $\int_C \vec{F}\cdot\vec{dr}$, where C is the boundary of the Portion of $z = 4 - x^2 - y^2$ above the xy-plane oriented upward and $\vec{F}(x,y,z) = (x^2e^x - y)\,\vec{i} + \sqrt{y^2+1}\,\vec{j} + z^3\,\vec{k}$.

Solution: We find $curl\left(\vec{F}\right)=\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2e^x-y & \sqrt{y^2+1} & z^3 \end{bmatrix}$

$$=\vec{i}(0)-\vec{j}(0)+\vec{k}(-1)=(M_1\vec{i}+N_1\vec{j}+P_1\vec{k})$$

$$z=4-x^2-y^2=g(x,y)\Rightarrow g_x=-2x, g_y=-2y$$

$$\Rightarrow \iint_S curl\left(\vec{F}\right)\bullet\vec{n}\,dS=\iint_R (-M_1g_x-N_1g_x+P_1)dA=\iint_R (0+0+1)dA=dA$$

$$\Rightarrow \int_0^{2\pi}\int_0^2 dA= \int_0^{2\pi}\int_0^2 rdrd\theta=4\pi$$

(d) Verify the divergence theorem by evaluating both the surface integral and the Triple integral for the function $\vec{F}(x,y,z)=x\vec{i}-y\vec{j}+z\vec{k}$, and the surface S which is the solid bounded by the graphs of $z=x^2+y^2$ and $z=4$.

Solution: Here we have to prove $\iint_S \vec{F}\bullet\vec{n}\,dS=\iiint_Q\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial z}{\partial z}\right)dV$.......(1)

L.H.S=R.H.S..(1)

First we calculate L.H.S. of (1) i.e. $\iiint_Q\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial z}{\partial z}\right)dA.=\iiint_Q (1-1+1)dV$

$$=\iiint_Q dV$$

$$=\int_0^{2\pi}\int_0^2\int_{r^2}^4 rdzdrd\theta=8\pi$$

Now we evaluate the L.H.S. of (1) i.e. $\iint_S \vec{F}\bullet\vec{n}\,dS$

The surface S is divided into two parts namely:

S_1: Plane $z=4=g_1(x,y)$ *this portion has upper normal vector*

S_2: Paraboloid $z=x^2+y^2=g_2(x,y)$ this portion has lower normal.

Hence, $\iint_S \vec{F}\bullet\vec{n}\,dS=\iint_{S_1}\vec{F}\bullet\vec{n}\,dS_1+\iint_{S_2}\vec{F}\bullet\vec{n}\,dS_2$

$$=\iint_{R_{xy}}(-Mg_{1x}-Ng_{1y}+P)dA+\iint_{R_{xy}}Mg_{2x}+Ng_{2y}-P)\ dA$$

$$=\iint_{R_{xy}}(-Mg_{1x}-Ng_{1y}+P)dA+\iint_{R_{xy}}Mg_{2x}+Ng_{2y}-P)\ dA$$

$$=\iint_{R_{xy}}(-Mg_{1x}-Ng_{1y}+P)dA+\iint_{R_{xy}}Mg_{2x}+Ng_{2y}-P)\ dA$$

$$=\iint_{R_{xy}}(-x(0)-(-y)(0)+z)dA+\iint_{R_{xy}}2x^2-2y^2-z)\ dA$$

$$= \iint\limits_{R_{xy}} z dA + \iint\limits_{R_{xy}} (2x^2 - 2y^2 - x^2 - y^2) dA$$

$$= \int_0^{2\pi} \int_0^{2} r dr d\theta + \iint\limits_{R_{xy}} (x^2 - 3y^2) dA$$

$$= 16\pi + \int_0^{2\pi} \int_0^{2} (r^2 Cos^2\theta - 3r^2 Sin^2\theta) r dr d\theta = 16\pi - 8\pi = 8\pi$$