Discrete Mathematics (151)

Department of Mathematics
College of Sciences
King Saud University

Chapter 1:
The Foundations: Logic and Proofs

1.1 Propositional Logic

Propositions

Our discussion begins with an introduction to the basic building blocks of logic-propositions. A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

EXAMPLE 1

All the following declarative sentences are propositions.

1. $\sqrt{2}$ is a real number.
2. -5 is a positive integer.
3. $2 > 4$.
4. $1 + 2 = 3$.

Propositions 1 and 4 are true, whereas 2 and 3 are false.

EXAMPLE 2

Consider the following sentences.

1. What times is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

EXAMPLE 2

Consider the following sentences.

1. What times is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

Propositional Logic

- We use letters to denote **propositional variables** (or **statement variables**). The conventional letters used for propositional variables are p, q, r, s, . . .
- The **truth value** of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.
- The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**.
- We now turn our attention to methods for producing new propositions from those that we already have. Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

DEFINITION 1

Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\bar{p}$), is the statement "It is not the case that p."
The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

EXAMPLE 3

Find the negations of the following propositions:

1. $2 = 3$;
2. $6 \leq 4$;
3. $2 \geq -2$;
4. $2 < 0$;
5. $3 > 2$.

Solution: The negations are:

1. $2 \neq 3$;
2. $6 > 4$;
3. $2 < -2$;
4. $2 \geq 0$;
5. $3 \leq 2$.

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Solution: The negations are:

1. $2 \neq 3$;
2. $6 > 4$;
3. $2 < -2$;
4. $2 \geq 0$;
5. $3 \leq 2$.

EXAMPLE 4

Find the negations of the following propositions

1. " n is an integer".
2. " n is a negative integer".

Solution:

1. " n is not an integer".
2. " n is a non negative integer".

Truth Table

TABLE 1	
p	$\neg p$
T	F
F	T

Table 1 displays the **truth table** for the negation of a proposition p. This table has a row for each of the two possible truth values of a proposition p. Each row shows the truth value of $\neg p$ corresponding to the truth value of p for this row.

EXAMPLE 4

Find the negations of the following propositions

1. " *n* is an integer".
2. " *n* is a negative integer".

Solution:

1. " *n* is not an integer".
2. " *n* is a non negative integer".

Truth Table

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p	$\neg p$
T	F
F	T

Table 1 displays the **truth table** for the negation of a proposition p. This table has a row for each of the two possible truth values of a proposition p. Each row shows the truth value of $\neg p$ corresponding to the truth value of p for this row.

DEFINITION 2

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

EXAMPLE 5

Find the conjunction of the propositions p and q where p is the proposition " $2 < 5$" and q is the proposition " $2 \geqslant -6$."

Solution: The conjunction of these propositions, $p \wedge q$, is the proposition " $2 < 5$ and $2 \geqslant -6$."

This conjunction can be expressed more simply as " $-6 \leqslant 2 < 5$."

For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

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For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

DEFINITION 3

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

EXAMPLE 6

What is the disjunction of the propositions p and q where p is the proposition "$-3 \in \mathbb{R}$" and q is the proposition " $-3 \in \mathbb{N}$."

Solution: The disjunction of p and q, $p \vee q$, is the proposition "$-3 \in \mathbb{R}$ or $-3 \in \mathbb{N}$"

This proposition is true.

DEFINITION 3

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

EXAMPLE 6

What is the disjunction of the propositions p and q where p is the proposition "$-3 \in \mathbb{R}$" and q is the proposition " $-3 \in \mathbb{N}$."
Solution: The disjunction of p and q, $p \vee q$, is the proposition "$-3 \in \mathbb{R}$ or $-3 \in \mathbb{N}$"
This proposition is true.

Truth Table

TABLE 2		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2 displays the truth table of $p \wedge q$.

TABLE 3		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 3 displays the truth table of $p \vee q$.

DEFINITION 4

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Conditional Statements

We will discuss several other important ways in which propositions can be combined.

DEFINITION 5

Let p and q be propositions. The conditional statement $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

Truth Table

TABLE 4		
p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 4 displays the truth table of $p \oplus q$.

TABLE 5		
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 5 displays the truth table of $p \rightarrow q$.

- In the conditional statement $p \to q$, p is called the **hypothesis** (or **antecedent** or **premise**) and q is called the **conclusion** (or **consequence**).
- The statement $p \to q$ is called a conditional statement because $p \to q$ asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.
- the statement $p \to q$ is true when both p and q are true and when p is false (no matter what truth value q has).
- Conditional statements play such an essential role in mathematical reasoning.

Terminology is used to express $p \to q$.

"if p, then q"	" p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless $\neg p$"	

CONVERSE, CONTRAPOSITIVE, AND INVERSE

We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names.

- The proposition $q \to p$ is called the **converse** of $p \to q$.
- The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$.
- The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$.

We will see that of these three conditional statements formed from $p \to q$, only the contrapositive always has the same truth value as $p \to q$.

EXAMPLE 7

What are the contrapositive, the converse, and the inverse of the conditional statement " $\sqrt{x}$ exist whenever the real x is positive."?

Solution: Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as "If the real x is positive, then $\sqrt{x}$ exist"

Consequently, the contrapositive is "If $\sqrt{x}$ does not exist, the real x is not positive, then"

The converse is "$\sqrt{x}$ exist, then the real x is positive."

The inverse is "If the real x is not positive, then $\sqrt{x}$ does not exist"

Only the contrapositive is equivalent to the original statement.

EXAMPLE 7

What are the contrapositive, the converse, and the inverse of the conditional statement " $\sqrt{x}$ exist whenever the real x is positive."?

Solution: Because "q whenever p" is one of the ways to express the conditional statement $p \rightarrow q$, the original statement can be rewritten as "If the real x is positive, then $\sqrt{x}$ exist"

Consequently, the contrapositive is "If $\sqrt{x}$ does not exist, the real x is not positive, then"

The converse is "$\sqrt{x}$ exist, then the real x is positive."

The inverse is "If the real x is not positive, then $\sqrt{x}$ does not exist"

Only the contrapositive is equivalent to the original statement.

BICONDITIONALS

DEFINITION 6

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q". **The biconditional statement $p \leftrightarrow q$** is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called **bi-implications.**

Note that the statement $p \leftrightarrow q$ is true when both the conditional statements $p \rightarrow q$ and $q \rightarrow p$ are true and is false otherwise. That is why we use the words "if and only if" to express this logical connective and why it is symbolically written by combining the symbols $\rightarrow$ and $\leftarrow$.

Propositional Logic

There are some other common ways to express $p \leftrightarrow q$:

- "p is necessary and sufficient for q"
- "if p then q, and conversely"
- "p iff q."

The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation "iff" for "if and only if." Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$.

Truth Table

TABLE 6		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 6 displays the truth table of $p \leftrightarrow q$.

EXAMPLE 8 (10 in book)

Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement "You can take the flight if and only if you buy a ticket."
This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values.

Truth Tables of Compound Propositions

- We have now introduced four important logical connectives: conjunctions, disjunctions, conditional statements, and biconditional statements, as well as negations.
- We can use these connectives to build up complicated compound propositions involving any number of propositional variables.
- We can use truth tables to determine the truth values of these compound propositions.

EXAMPLE 9 (11 in book)

Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$.

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

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Construct the truth table of the compound proposition $(p \vee \neg q) \to (p \wedge q)$.

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p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

1.2 Propositional Equivalences (1.3 in book)

Propositional Equivalences

Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value.
Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments.

DEFINITION 1

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Example 1

We can construct examples of tautologies and contradictions using just one propositional variable.
Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction.

Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called **logically equivalent**. We can also define this notion as follows.

DEFINITION 2

The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology.
The notation $p \equiv q$ denotes that p and q are logically equivalent.

TABLE 1: Examples of a Tautology and a Contradiction.

TABLE 1			
p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

TABLE 2: De Morgan's Laws.

TABLE 2
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example 2

1. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.
2. Show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table 3.

TABLE 3 The Truth Table

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg$
T	T	T	F	F	F	F	T	F	F
T	F	T	F	F	T	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	F	T	T	T	T	F	T	T

Propositional Equivalences

Example 2

1. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.
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Solution: We construct the truth table for these compound propositions in Table 3.

TABLE 3 The Truth Table

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg$
T	T	T	F	F	F	F	T	F	F
T	F	T	F	F	T	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	F	T	T	T	T	F	T	T

Example 3

1. Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.
2. Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table 4.

TABLE 4 The Truth Table

p	q	$\neg p$	$\neg q$	$\neg p \vee q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Example 3

1. Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.
2. Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table 4.

TABLE 4 The Truth Table

p	q	$\neg p$	$\neg q$	$\neg p \vee q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Example 4

Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the **distributive law** of disjunction over conjunction.

Solution: We construct the truth table for these compound propositions in Table 5.

TABLE 5 The Truth Table

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F

Example 4

Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.
This is the **distributive law** of disjunction over conjunction.
Solution: We construct the truth table for these compound propositions in Table 5.

TABLE 5 The Truth Table

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F

Propositional Equivalences

TABLE 6 Logical Equivalences	
Equivalence	Name
$p \wedge T \equiv p$, $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$, $p \wedge F \equiv F$	Domination laws
$p \wedge p \equiv p$, $p \vee p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \wedge q \equiv q \wedge p$ and $p \vee q \equiv q \vee p$	Commutative laws
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$	Negation laws

Propositional Equivalences

Table 7: Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \wedge r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Table 8: Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

Example 5 (6 in book)

Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution: We could use a truth table to show that these compound propositions are equivalent.

So, we will establish this equivalence by developing a series of logical equivalences, using one of the equivalences in Table 6 at a time, starting with $\neg(p \rightarrow q)$ and ending with $p \wedge \neg q$.

We have the following equivalences.

$\neg(p \rightarrow q \equiv \neg(\neg p \vee q)$ by Example 3

$\equiv \neg(\neg p) \wedge \neg q$ by the second De Morgan's law

$\equiv p \wedge \neg q$ by the double negation law

Constructing New Logical Equivalences

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Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

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$\equiv \neg(\neg p) \wedge \neg q$ by the second De Morgan's law

$\equiv p \wedge \neg q$ by the double negation law

Example 6 (7 in book)

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: Solution: We will use one of the equivalences in Table 6 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$.

We have the following equivalences.

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ by the second De Morgan's laws

$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$ by the first De Morgan's laws

$\equiv \neg p \wedge (p \vee \neg q)$ by the double negation law

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ by the distributive laws

$\equiv F \vee (\neg p \neg q)$ because $\neg p \vee p \equiv F$

$\equiv (\neg p \wedge \neg q) \vee F$ by the commutative laws

$\equiv \neg p \wedge \neg q$ by the identity laws

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Propositional Equivalences

Example 6 (7 in book)

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: Solution: We will use one of the equivalences in Table 6 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$.

We have the following equivalences.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan's laws} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan's laws} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the distributive laws} \\
&\equiv F \vee (\neg p \neg q) && \text{because } \neg p \vee p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative laws} \\
&\equiv \neg p \wedge \neg q && \text{by the identity laws}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Example 7 (8 in book)

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **T**. (Note: This could also be done using a truth table.)

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by Example 3} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan's law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative laws for disjunction} \\
&\equiv T \vee T && \text{by Example 1 and commutative laws for disjunction} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$

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Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **T**. (Note: This could also be done using a truth table.)

$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ by Example 3

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ by the first De Morgan's law

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ by the associative and commutative laws for disjunction

$\equiv T \vee T$ by Example 1 and commutative laws for disjunction

$\equiv T$ by the domination law

1.3 Predicates and Quantifiers (1.4 in book)

Predicates

Statements involving variables, such as "$x > 3$", "$x = y + 3$", "$x + y = z$",

Example 1

Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

Solution: We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$." Hence, $P(4)$, which is the statement "$4 > 3$" is true. However, $P(2)$, which is the statement "$2 > 3$," is false.

Example 2 (3 in book)

Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Solution: To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "$1 = 2 + 3$," which is false. The statement $Q(3, 0)$ is the proposition "$3 = 0 + 3$," which is true.

Predicates

Statements involving variables, such as "$x > 3$", "$x = y + 3$", "$x + y = z$",

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Example 3 (5 in book)

Let $R(x, y, z)$ denote the statement "$x + y = z$", What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$?

Solution: The proposition $R(1, 2, 3)$ is obtained by setting $x = 1, y = 2$, and $z = 3$ in the statement $R(x, y, z)$. We see that $R(1, 2, 3)$ is the statement "$1 + 2 = 3$", which is true. Also note that $R(0, 0, 1)$, which is the statement "$0 + 0 = 1$", is false.

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Quantifiers

DEFINITION 1: THE UNIVERSAL QUANTIFIER

The universal quantification of P(x) is the statement

"P(x) for all values of x in the domain."

The notation $\forall x\ P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier**. We read $\forall x\ P(x)$ as "for all x $P(x)$" or "for every x $P(x)$". An element for which $P(x)$ is false is called a **counterexample** of $\forall x\ P(x)$.

DEFINITION 2: THE EXISTENTIAL QUANTIFIER

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$".

We use the notation $\exists x\ P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

- The statement $\forall x\ P(x)$ is true when $P(x)$ is true for every x and is false when there is an x for which $P(x)$ is false.
- The statement $\exists x\ P(x)$ is true when there is an x for which $P(x)$ is true and is false when $P(x)$ is false for every x.

$$\neg\,(\exists x\ Q(x)) \equiv \forall x\ \neg Q(x).$$

$$\neg\,(\forall x\ P(x)) \equiv \exists x\ \neg P(x).$$

Example 4 (8 in book)

Let $P(x)$ be the statement "$x + 1 > x$". What is the truth value of the quantification $\forall x \ P(x)$, where the domain consists of all real numbers?

Solution: Because $P(x)$ is true for all real numbers x, the quantification $\forall x \ P(x)$ is true.

Example 5 (9 in book)

Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x \ Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x \ Q(x)$. Thus $\forall x \ Q(x)$ is false.

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Predicates and Quantifiers

Example 6 (10 in book)

Suppose that $P(x)$ is "$x^2 > 0$." To show that the statement $\forall x\, P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a counterexample because $x^2 = 0$ when $x = 0$, so that x^2 is not greater than 0 when $x = 0$.

Looking for counterexamples to universally quantified statements is an important activity in the study of mathematics, as we will see in subsequent sections.

When all the elements in the domain can be listed–say, $x_1, x_2, ..., x_n$–it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjunction, $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$, because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

Example 7 (11 in book)

What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution: The statement $\forall x\, P(x)$ is the same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, because the domain consists of the integers $1, 2, 3$, and 4. Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x\, P(x)$ is false.

Example 8 (13 in book)

What is the truth value of $\forall x\,(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution: The universal quantification $\forall x\,(x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $\left(\frac{1}{2}\right)^2 \ngeq \frac{1}{2}$. Note that $x^2 \geq x$ if and only if $x^2 - x = x(x-1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x\,(x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$). However, if the domain consists of the integers, $\forall x\,(x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$.

Predicates and Quantifiers

TABLE 1 De Morgan's Laws for Quantifiers. (2 in book)

Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg(\exists x\, P(x))$	$\forall x\, \neg P(x)$	For every x, $P(x)$ is false.	There is an x for $P(x)$ which is true.
$\neg(\forall x\, P(x))$	$\exists x\, \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Example 9 (21 in book)

What are the negations of the statements $\forall x\, (x^2 > x)$ and $\exists x\, (x^2 = 2)$?

Solution: The negation of $\forall x\, (x^2 > x)$ is the statement $\neg\forall x\, (x^2 > x)$, which is equivalent to $\exists x\, \neg(x^2 > x)$. This can be rewritten as $\exists x\, (x2 \leq x)$.

The negation of $\exists x\, (x^2 = 2)$ is the statement $\neg\exists x\, (x^2 = 2)$, which is equivalent to $\forall x\, \neg(x^2 = 2)$. This can be rewritten as $\forall x\, (x^2 \neq 2)$.

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Example 10 (22 in book)

Show that $\neg\forall x\,(P(x) \to Q(x))$ and $\exists x\,(P(x) \land \neg Q(x))$ are logically equivalent.

Solution: By De Morgans law for universal quantifiers, we know that $\neg\forall x\,(P(x) \to Q(x))$ and $\exists x\,(\neg(P(x) \to Q(x)))$ are logically equivalent. By the fifth logical equivalence in Table 7 in Section 1.2 (1.3 in book), we know that $\neg(P(x) \to Q(x))$ and $P(x) \land \neg Q(x)$ are logically equivalent for every x. Because we can substitute one logically equivalent expression for another in a logical equivalence, it follows that $\neg\forall x\,(P(x) \to Q(x))$ and $\exists x\,(P(x) \land \neg Q(x))$ are logically equivalent.

Example 10 (22 in book)

Show that $\neg\forall x\,(P(x) \to Q(x))$ and $\exists x\,(P(x) \wedge \neg Q(x))$ are logically equivalent.

Solution: By De Morgans law for universal quantifiers, we know that $\neg\forall x\,(P(x) \to Q(x))$ and $\exists x\,(\neg(P(x) \to Q(x)))$ are logically equivalent. By the fifth logical equivalence in Table 7 in Section 1.2 (1.3 in book), we know that $\neg(P(x) \to Q(x))$ and $P(x) \wedge \neg Q(x)$ are logically equivalent for every x. Because we can substitute one logically equivalent expression for another in a logical equivalence, it follows that $\neg\forall x\,(P(x) \to Q(x))$ and $\exists x\,(P(x) \wedge \neg Q(x))$ are logically equivalent.

1.4 Introduction to Proofs (1.7 in book)

Methods of Proving Theorems

Proving mathematical theorems can be difficult. To construct proofs we need all available ammunition, including a powerful battery of different proof methods. These methods provide the overall approach and strategy of proofs. Understanding these methods is a key component of learning how to read and construct mathematical proofs. One we have chosen a proof method, we use axioms, definitions of terms, previously proved results, and rules of inference to complete the proof.

Direct Proofs

A **direct proof** of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true.

A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs. In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.

DEFINITION 1

The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k such that $n = 2k + 1$. (Note that every integer is either even or odd, and no integer is both even and odd.)
Two integers have the same parity when both are even or both are odd; they have opposite parity when one is even and the other is odd.

Introduction to Proofs

Example 1

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: Note that this theorem states $\forall n\,(P(n) \rightarrow Q(n))$, where $P(n)$ is "n is an odd integer" and $Q(n)$ is "n^2 is odd." To begin a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that n is odd. By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer. We want to show that n^2 is also odd. We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2. When we do this, we find that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. We can conclude that n^2 is an odd integer (it is one more than twice an integer). Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer.

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Example 2

Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a perfect square if there is an integer b such that $a = b^2$.)

Solution: To produce a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that m and n are both perfect squares. By the definition of a perfect square, it follows that there are integers s and t such that $m = s^2$ and $n = t^2$. The goal of the proof is to show that mn must also be a perfect square when m and n are; looking ahead we see how we can show this by substituting s^2 for m and t^2 for n into mn. This tells us that $mn = s^2t^2$. Hence, $mn = s^2t^2 = (ss)(tt) = (st)(st) = (st)^2$, using commutativity and associativity of multiplication. By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of st, which is an integer. We have proved that if m and n are both perfect squares, then mn is also a perfect square.

Example 2

Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a perfect square if there is an integer b such that $a = b^2$.)

Solution: To produce a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that m and n are both perfect squares. By the definition of a perfect square, it follows that there are integers s and t such that $m = s^2$ and $n = t^2$. The goal of the proof is to show that mn must also be a perfect square when m and n are; looking ahead we see how we can show this by substituting s^2 for m and t^2 for n into mn. This tells us that $mn = s^2t^2$. Hence, $mn = s^2t^2 = (ss)(tt) = (st)(st) = (st)^2$, using commutativity and associativity of multiplication. By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of st, which is an integer. We have proved that if m and n are both perfect squares, then mn is also a perfect square.

Proof by Contraposition

We need other methods of proving theorems of the form $\forall x\,(P(x) \rightarrow Q(x))$. Proofs of theorems of this type that are not direct proofs, that is, that do not start with the premises and end with the conclusion, are called **indirect proofs**.

An extremely useful type of indirect proof is known as **proof by contraposition**. Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$. This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true. In a proof by contraposition of $p \rightarrow q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow.

Example 3

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution: We first attempt a direct proof. To construct a direct proof, we first assume that $3n + 2$ is an odd integer. This means that $3n + 2 = 2k + 1$ for some integer k. Can we use this fact to show that n is odd? We see that $3n + 1 = 2k$, but there does not seem to be any direct way to conclude that n is odd. Because our attempt at a direct proof failed, we next try a proof by contraposition. The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then n is odd" is false; namely, assume that n is even. Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd. This is the negation of the premise of the theorem. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem "If $3n + 2$ is odd, then n is odd."

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Example 4

Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution: Because there is no obvious way of showing that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ directly from the equation $n = ab$, where a and b are positive integers, we attempt a proof by contraposition.

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$" is false. That is, we assume that the statement $(a \leq \sqrt{n}) \vee (b \leq \sqrt{n})$ is false. Using the meaning of disjunction together with De Morgans law, we see that this implies that both $a \leq \sqrt{n}$ and $b \leq \sqrt{n}$ are false. This implies that $a > \sqrt{n}$ and $b > \sqrt{n}$. We can multiply these inequalities together (using the fact that if $0 < s < t$ and $0 < u < v$, then $su < tv$) to obtain $ab > \sqrt{n}\sqrt{n} = n$. This shows that $ab \neq n$, which contradicts the statement $n = ab$. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Example 4

Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.
Solution: Because there is no obvious way of showing that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ directly from the equation $n = ab$, where a and b are positive integers, we attempt a proof by contraposition.
The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$" is false. That is, we assume that the statement $(a \leq \sqrt{n}) \vee (b \leq \sqrt{n})$ is false. Using the meaning of disjunction together with De Morgans law, we see that this implies that both $a \leq \sqrt{n}$ and $b \leq \sqrt{n}$ are false. This implies that $a > \sqrt{n}$ and $b > \sqrt{n}$. We can multiply these inequalities together (using the fact that if $0 < s < t$ and $0 < u < v$, then $su < tv$) to obtain $ab > \sqrt{n}\sqrt{n} = n$. This shows that $ab \neq n$, which contradicts the statement $n = ab$. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Example 5 (8 in book)

Prove that if n is an integer and n^2 is odd, then n is odd.

Solution: We first attempt a direct proof. Suppose that n is an integer and n^2 is odd. Then, there exists an integer k such that $n^2 = 2k + 1$. Can we use this information to show that n is odd?

There seems to be no obvious approach to show that n is odd because solving for n produces the equation $n = \pm\sqrt{2k + 1}$, which is not terribly useful. Because this attempt to use a direct proof did not give result, we next attempt a proof by contraposition. We take as our hypothesis the statement that n is not odd. Because every integer is odd or even, this means that n is even. This implies that there exists an integer k such that $n = 2k$. To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd, that is, that n^2 is even. Can we use the equation $n = 2k$ to achieve this? By squaring both sides of this equation, we obtain $n^2 = 4k^2 = 2(2k^2)$, which implies that n^2 is also even because $n^2 = 2t$, where $t = 2k^2$. We have proved that if n is an integer and n^2 is odd, then n is odd. Our attempt to find a proof by contraposition succeeded.

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Proofs by Contradiction

Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \rightarrow q$ is true. Because q is false, but $\neg p \rightarrow q$ is true, we can conclude that $\neg p$ is false, which means that p is true. How can we find a contradiction q that might help us prove that p is true in this way?

Because the statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \rightarrow (r \wedge \neg r)$ is true for some proposition r. Proofs of this type are called proofs by contradiction. Because a proof by contradiction does not prove a result directly, it is another type of indirect proof.

Example 6 (9 in book)

Show that at least four of any 22 days must fall on the same day of the week.

Solution: Let p be the proposition "At least four of 22 chosen days fall on the same day of the week". Suppose that $\neg p$ is true. This means that at most three of the 22 days fall on the same day of the week. Because there are seven days of the week, this implies that at most 21 days could have been chosen, as for each of the days of the week, at most three of the chosen days could fall on that day. This contradicts the premise that we have 22 days under consideration. That is, if r is the statement that 22 days are chosen, then we have shown that $\neg p \to (r \wedge \neg r)$. Consequently, we know that p is true. We have proved that at least four of 22 chosen days fall on the same day of the week.

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Example 7 (10 in book)

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution: Let p be the proposition "$\sqrt{2}$ is irrational." To start a proof by contradiction, we supposethat $\neg p$ is true. Note that $\neg p$ is the statement "It is not the case that $\sqrt{2}$ is irrational," which says that $\sqrt{2}$ is rational. We will show that assuming that $\neg p$ is true leads to a contradiction. If $\sqrt{2}$ is rational, there exist integers a and b with $\sqrt{2} = \frac{a}{b}$, where $b \neq 0$ and a and b have no common factors (so that the fraction $\frac{a}{b}$ is in lowest terms.) (Here, we are using the fact that every rational number can be written in lowest terms.) Because $\sqrt{2} = \frac{a}{b}$, when both sides of this equation are squared, it follows that $2 = \frac{a^2}{b^2}$. Hence, $2b^2 = a^2$. By the definition of an even integer it follows that a^2 is even. We next use the fact that if a^2 is even, a must also be even.

Example 7 (10 in book)

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Furthermore, because a is even, by the definition of an even integer, $a = 2c$ for some integer c. Thus, $2b^2 = 4c^2$. Dividing both sides of this equation by 2 gives $b^2 = 2c^2$. By the definition of even, this means that b^2 is even. Again using the fact that if the square of an integer is even, then the integer itself must be even, we conclude that b must be even as well. We have now shown that the assumption of $\neg p$ leads to the equation $\sqrt{2} = \frac{a}{b}$, where a and b have no common factors, but both a and b are even, that is, 2 divides both a and b. Note that the statement that $\sqrt{2} = \frac{a}{b}$, where a and b have no common factors, means, in particular, that 2 does not divide both a and b. Because our assumption of $\neg p$ leads to the contradiction that 2 divides both a and b and 2 does not divide both a and b, $\neg p$ must be false. That is, the statement p, "$\sqrt{2}$ is irrational," is true. We have proved that $\sqrt{2}$ is irrational.

Example 8 (11 in book)

Give a proof by contradiction of the theorem "If $3n+2$ is odd, then n is odd."

Solution: Let p be "$3n+2$ is odd" and q be "n is odd." To construct a proof by contradiction, assume that both p and $\neg q$ are true. That is, assume that $3n+2$ is odd and that n is not odd. Because n is not odd, we know that it is even. Because n is even, there is an integer k such that $n = 2k$. This implies that $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$. Because $3n+2$ is $2t$, where $t = 3k+1$, $3n+2$ is even. Note that the statement "$3n+2$ is even" is equivalent to the statement $\neg p$, because an integer is even if and only if it is not odd. Because both p and $\neg p$ are true, we have a contradiction. This completes the proof by contradiction, proving that if $3n+2$ is odd, then n is odd.

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Note that we can also prove by contradiction that $p \to q$ is true by assuming that p and $\neg q$ are true, and showing that q must be also be true. This implies that $\neg q$ and q are both true, a contradiction. This observation tells us that we can turn a direct proof into a proof by contradiction.

Proofs of Equivalence

To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we show that $p \to q$ and $q \to p$ are both true. The validity of this approach is based on the tautology
$(p \leftrightarrow q) \leftrightarrow (p \to q) \wedge (q \to p)$.

Example 9 (12 in book)

Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd."

Solution: This theorem has the form "p if and only if q," where p is "n is odd" and q is "n^2 is odd." (As usual, we do not explicitly deal with the universal quantification.) To prove this theorem, we need to show that $p \to q$ and $q \to p$ are true. We have already shown (in Example 1) that $p \to q$ is true and (in Example 5 (8 in book)) that $q \to p$ is true. Because we have shown that both $p \to q$ and $q \to p$ are true, we have shown that the theorem is true.

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1.5 Proof Methods and Strategy (1.8 in book)

Proof Methods and Strategy

Introduction

- In Section 1.4 (1.7 in book) we introduced many methods of proof and illustrated how each method can be used. In this section we continue this effort. We will introduce several other commonly used proof methods, including the method of proving a theorem by considering different cases separately. We will also discuss proofs where we prove the existence of objects with desired properties.
- In Section 1.4 (1.7 in book) we briefly discussed the strategy behind constructing proofs. This strategy includes selecting a proof method and then successfully constructing an argument step by step, based on this method.

Introduction

- In this section, after we have developed a versatile arsenal of proof methods, we will study some aspects of the art and science of proofs.
- We will provide advice on how to find a proof of a theorem. We will describe some tricks of the trade, including how proofs can be found by working backward and by adapting existing proofs.
- When mathematicians work, they formulate conjectures and attempt to prove or disprove them.

Exhaustive Proofs

- Some theorems can be proved by examining a relatively small number of examples. Such proofs are called **exhaustive proofs**, or **proofs by exhaustion** because these proofs proceed by exhausting all possibilities.
- An exhaustive proof is a special type of proof by cases where each case involves checking a single example.
- We now provide some illustrations of exhaustive proofs.
  To prove a conditional statement of the form $(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q$ the tautology
  $[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \vee (p_n \rightarrow q)]$
  can be used as a rule of inference.

Example 1

Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Solution: We use a proof by exhaustion. We only need verify the inequality $(n+1)^3 \geq 3^n$ when $n = 1, 2, 3$, and 4.
For $n = 1$, we have $(n+1)^3 = 2^3 = 8$ and $3^n = 3^1 = 3$;
for $n = 2$, we have $(n+1)^3 = 3^3 = 27$ and $3^n = 3^2 = 9$;
for $n = 3$, we have $(n+1)^3 = 4^3 = 64$ and $3^n = 3^3 = 27$;
and for $n = 4$, we have $(n+1)^4 = 5^4 = 625$ and $3^n = 3^4 = 81$.
In each of these four cases, we see that $(n+1)^3 \geq 3^n$. We have used the method of exhaustion to prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

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and for $n = 4$, we have $(n+1)^4 = 5^4 = 625$ and $3^n = 3^4 = 81$.

In each of these four cases, we see that $(n+1)^3 \geq 3^n$. We have used the method of exhaustion to prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Example 2

Prove that the only consecutive positive integers not exceeding 100 that are perfect powers are 8 and 9. (An integer is a **perfect power** if it equals n^a, where a is an integer greater than 1.)

Solution: We use a proof by exhaustion. In particular, we can prove this fact by examining positive integers n not exceeding 100, first checking whether n is a perfect power, and if it is, checking whether $n+1$ is also a perfect power. A quicker way to do this is simply to look at all perfect powers not exceeding 100 and checking whether the next largest integer is also a perfect power. The squares of positive integers not exceeding 100 are 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100. The cubes of positive integers not exceeding 100 are 1, 8, 27, and 64. The fourth powers of positive integers not exceeding 100 are 1, 16, and 81.

Example 2

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Proof Methods and Strategy

The fifth powers of positive integers not exceeding 100 are 1 and 32. The sixth powers of positive integers not exceeding 100 are 1 and 64. There are no powers of positive integers higher than the sixth power not exceeding 100, other than 1.
Looking at this list of perfect powers not exceeding 100, we see that $n = 8$ is the only perfect power n for which $n + 1$ is also a perfect power. That is, $2^3 = 8$ and $3^2 = 9$ are the only two consecutive perfect powers not exceeding 100.

Proof by Cases

- A proof by cases must cover all possible cases that arise in a theorem.
- We illustrate proof by cases with a couple of examples. In each example, you should check that all possible cases are covered.

Example 3

Prove that if n is an integer, then $n^2 \geq n$.

Solution: We can prove that $n^2 \geq n$ for every integer by considering three cases, when $n = 0$, when $n \geq 1$, and when $n \leq -1$. We split the proof into three cases because it is straightforward to prove the result by considering zero, positive integers, and negative integers separately.

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Proof Methods and Strategy

- Case (i): When $n = 0$, because $0^2 = 0$, we see that $0^2 \geq 0$. It follows that $n^2 \geq n$ is true in this case.
- Case (ii): When $n \geq 1$, when we multiply both sides of the inequality $n \geq 1$ by the positive integer n, we obtain $n.n \geq n.1$. This implies that $n^2 \geq n$ for $n \geq 1$.
- Case (iii): In this case $n \leq -1$. However, $n^2 \geq 0$. It follows that $n^2 \geq n$.

Because the inequality $n^2 \geq n$ holds in all three cases, we can conclude that if n is an integer, then $n^2 \geq n$.

Proof Methods and Strategy

Example 4

Use a proof by cases to show that $|xy| = |x||y|$, where x and y are real numbers. (Recall that $|a|$, the absolute value of a, equals a when $a \geq 0$ and equals $-a$ when $a \leq 0$.)

Solution: In our proof of this theorem, we remove absolute values using the fact that $|a| = a$ when $a \geq 0$ and $|a| = -a$ when $a < 0$. Because both $|x|$ and $|y|$ occur in our formula, we will need four cases: (i) x and y both nonnegative, (ii) x nonnegative and y is negative, (iii) x negative and y nonnegative, and (iv) x negative and y negative. We denote by p_1, p_2, p_3, and p_4, the proposition stating the assumption for each of these four cases, respectively.

(Note that we can remove the absolute value signs by making the appropriate choice of signs within each case.)

Example 4

Use a proof by cases to show that $|xy| = |x||y|$, where x and y are real numbers. (Recall that $|a|$, the absolute value of a, equals a when $a \geq 0$ and equals $-a$ when $a \leq 0$.)

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(Note that we can remove the absolute value signs by making the appropriate choice of signs within each case.)

- Case (i): We see that $p_1 \to q$ because $xy \geq 0$ when $x \geq 0$ and $y \geq 0$, so that $|xy| = xy = |x||y|$.
- To see that $p_2 \to q$, note that if $x \geq 0$ and $y < 0$, then $xy \leq 0$, so that $|xy| = -xy = x(-y) = |x||y|$. (Here, because $y < 0$, we have $|y| = -y$.)
- Case (iii): To see that $p_3 \to q$, we follow the same reasoning as the previous case with the roles of x and y reversed.
- Case (iv): To see that $p_4 \to q$, note that when $x < 0$ and $y < 0$, it follows that $xy > 0$. Hence, $|xy| = xy = (-x)(-y) = |x||y|$.

Because $|xy| = |x||y|$ holds in each of the four cases and these cases exhaust all possibilities, we can conclude that $|xy| = |x||y|$, whenever x and y are real numbers.

Example 5 (7 in book)

Show that if x and y are integers and both xy and $x + y$ are even, then both x and y are even.

Solution: We will use proof by contraposition, the notion of without loss of generality, and proof by cases. First, suppose that x and y are not both even. That is, assume that x is odd or that y is odd (or both). Without loss of generality, we assume that x is odd, so that $x = 2m + 1$ for some integer m.

To complete the proof, we need to show that xy is odd or $x + y$ is odd. Consider two cases: (i) y even, and (ii) y odd.

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To complete the proof, we need to show that xy is odd or $x + y$ is odd. Consider two cases: (i) y even, and (ii) y odd.

- In (i), $y = 2n$ for some integer n, so that $x + y = (2m + 1) + 2n = 2(m + n) + 1$ is odd.
- In (ii), $y = 2n + 1$ for some integer n, so that $xy = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$ is odd.

This completes the proof by contraposition. (Note that our use of without loss of generality within the proof is justified because the proof when y is odd can be obtained by simply interchanging the roles of x and y in the proof we have given.)