Discrete Mathematics (151)

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Chapter 1: Logic

1.1 Propositional Logic

Propositional Logic

Propositions

Our discussion begins with an introduction to the basic building blocks of logic-propositions. A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

EXAMPLE 1

All the following declarative sentences are propositions.

1. $\sqrt{2}$ is a real number.
2. -5 is a positive integer.
3. $2 > 4$.
4. $1 + 2 = 3$.

Propositions 1 and 4 are true, whereas 2 and 3 are false.

EXAMPLE 2

Consider the following sentences.

1. What times is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

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Propositional Logic

- We use letters to denote **propositional variables** (or **statement variables**). The conventional letters used for propositional variables are p, q, r, s, ...
- The **truth value** of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.
- The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**.
- We now turn our attention to methods for producing new propositions from those that we already have. Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

Propositional Logic

DEFINITION 1

Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\bar{p}$), is the statement "It is not the case that p."
The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

EXAMPLE 3

Find the negations of the following propositions:

1. $2 = 3$;
2. $6 \leq 4$;
3. $2 \geq -2$;
4. $2 < 0$;
5. $3 > 2$.

Solution: The negations are:

1. $2 \neq 3$;
2. $6 > 4$;
3. $2 < -2$;
4. $2 \geq 0$;
5. $3 \leq 2$.

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Solution: The negations are:

1. $2 \neq 3$;
2. $6 > 4$;
3. $2 < -2$;
4. $2 \geq 0$;
5. $3 \leq 2$.

EXAMPLE 4

Find the negations of the following propositions

1. " -1 is an integer".
2. " -1 is a negative integer".

Solution:

1. " -1 is not an integer".
2. " -1 is a non negative integer".

Truth Table

TABLE 1

p	$\neg p$
T	F
F	T

Table 1 displays the **truth table** for the negation of a proposition p. This table has a row for each of the two possible truth values of a proposition p. Each row shows the truth value of $\neg p$ corresponding to the truth value of p for this row.

EXAMPLE 4

Find the negations of the following propositions

1. " -1 is an integer".
2. " -1 is a negative integer".

Solution:

1. " -1 is not an integer".
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Propositional Logic

DEFINITION 2

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

EXAMPLE 5

Find the conjunction of the propositions p and q where p is the proposition " $2 < 5$" and q is the proposition " $2 \geqslant -6$."

Solution: The conjunction of these propositions, $p \wedge q$, is the proposition " $2 < 5$ and $2 \geqslant -6$."

This conjunction can be expressed more simply as " $-6 \leqslant 2 < 5$."

For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

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For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

DEFINITION 3

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

EXAMPLE 6

What is the disjunction of the propositions p and q where p is the proposition "$-3 \in \mathbb{R}$" and q is the proposition " $-3 \in \mathbb{N}$."

Solution: The disjunction of p and q, $p \vee q$, is the proposition "$-3 \in \mathbb{R}$ or $-3 \in \mathbb{N}$"

This proposition is true.

DEFINITION 3

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

EXAMPLE 6

What is the disjunction of the propositions p and q where p is the proposition "$-3 \in \mathbb{R}$" and q is the proposition " $-3 \in \mathbb{N}$."
Solution: The disjunction of p and q, $p \vee q$, is the proposition "$-3 \in \mathbb{R}$ or $-3 \in \mathbb{N}$"
This proposition is true.

Truth Table

TABLE 2		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2 displays the truth table of $p \wedge q$.

TABLE 3		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 3 displays the truth table of $p \vee q$.

DEFINITION 4

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Conditional Statements

We will discuss several other important ways in which propositions can be combined.

DEFINITION 5

Let p and q be propositions. The conditional statement $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

Propositional Logic

Truth Table

TABLE 4		
p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 4 displays the truth table of $p \oplus q$.

TABLE 5		
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 5 displays the truth table of $p \to q$.

Propositional Logic

- In the conditional statement $p \to q$, p is called the **hypothesis** (or **antecedent** or **premise**) and q is called the **conclusion** (or **consequence**).
- The statement $p \to q$ is called a conditional statement because $p \to q$ asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.
- the statement $p \to q$ is true when both p and q are true and when p is false (no matter what truth value q has).
- Conditional statements play such an essential role in mathematical reasoning.

Terminology is used to express $p \to q$.

"if p, then q"	" p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless $\neg p$"	

CONVERSE, CONTRAPOSITIVE, AND INVERSE

We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names.

- The proposition $q \to p$ is called the **converse** of $p \to q$.
- The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$.
- The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$.

We will see that of these three conditional statements formed from $p \to q$, only the contrapositive always has the same truth value as $p \to q$.

EXAMPLE 7

What are the contrapositive, the converse, and the inverse of the conditional statement " $\sqrt{2}$ exist whenever the real 2 is positive."?

Solution: Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as "If the real 2 is positive, then $\sqrt{2}$ exist"

Consequently, the contrapositive is "If $\sqrt{2}$ does not exist, the real 2 is not positive, then"

The converse is "$\sqrt{2}$ exist, then the real 2 is positive."

The inverse is "If the real 2 is not positive, then $\sqrt{2}$ does not exist"

Only the contrapositive is equivalent to the original statement.

EXAMPLE 7

What are the contrapositive, the converse, and the inverse of the conditional statement " $\sqrt{2}$ exist whenever the real 2 is positive."?

Solution: Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as "If the real 2 is positive, then $\sqrt{2}$ exist"

Consequently, the contrapositive is "If $\sqrt{2}$ does not exist, the real 2 is not positive, then"

The converse is "$\sqrt{2}$ exist, then the real 2 is positive."

The inverse is "If the real 2 is not positive, then $\sqrt{2}$ does not exist"

Only the contrapositive is equivalent to the original statement.

BICONDITIONALS

DEFINITION 6

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q". **The biconditional statement $p \leftrightarrow q$** is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called **bi-implications.**

Note that the statement $p \leftrightarrow q$ is true when both the conditional statements $p \rightarrow q$ and $q \rightarrow p$ are true and is false otherwise. That is why we use the words "if and only if" to express this logical connective and why it is symbolically written by combining the symbols $\rightarrow$ and $\leftarrow$.

Propositional Logic

There are some other common ways to express $p \leftrightarrow q$:

- "p is necessary and sufficient for q"
- "if p then q, and conversely"
- "p iff q."

The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation "iff" for "if and only if." Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$.

Truth Table

TABLE 6

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 6 displays the truth table of $p \leftrightarrow q$.

EXAMPLE 8 (10 in book)

Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement "You can take the flight if and only if you buy a ticket."
This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values.

Truth Tables of Compound Propositions

- We have now introduced four important logical connectives: conjunctions, disjunctions, conditional statements, and biconditional statements, as well as negations.
- We can use these connectives to build up complicated compound propositions involving any number of propositional variables.
- We can use truth tables to determine the truth values of these compound propositions.

EXAMPLE 9 (11 in book)

Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$.

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

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T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

1.2 Propositional Equivalences (1.3 in book)

Propositional Equivalences

Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value.
Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments.

DEFINITION 1

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Example 1

We can construct examples of tautologies and contradictions using just one propositional variable.
Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction.

Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called **logically equivalent**. We can also define this notion as follows.

DEFINITION 2

The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology.
The notation $p \equiv q$ denotes that p and q are logically equivalent.

Propositional Equivalences

TABLE 1: Examples of a Tautology and a Contradiction.

TABLE 1			
p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

TABLE 2: De Morgan's Laws.

TABLE 2
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example 2

1. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.
2. Show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table 3.

TABLE 3 The Truth Table

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee$
T	T	T	F	F	F	F	T	F	F
T	F	T	F	F	T	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	F	T	T	T	T	F	T	T

Propositional Equivalences

Example 2

1. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.
2. Show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

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TABLE 3 The Truth Table

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg$
T	T	T	F	F	F	F	T	F	F
T	F	T	F	F	T	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	F	T	T	T	T	F	T	T

Propositional Equivalences

Example 3

1. Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.
2. Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table 4.

TABLE 4 The Truth Table

p	q	$\neg p$	$\neg q$	$\neg p \vee q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

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2. Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table 4.

TABLE 4 The Truth Table

p	q	$\neg p$	$\neg q$	$\neg p \vee q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Example 4

Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the **distributive law** of disjunction over conjunction.

Solution: We construct the truth table for these compound propositions in Table 5.

TABLE 5 The Truth Table							
p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

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TABLE 5 The Truth Table

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Propositional Equivalences

TABLE 6 Logical Equivalences	
Equivalence	Name
$p \wedge T \equiv p$, $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$, $p \wedge F \equiv F$	Domination laws
$p \wedge p \equiv p$, $p \vee p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \wedge q \equiv q \wedge p$ and $p \vee q \equiv q \vee p$	Commutative laws
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$	Negation laws

Propositional Equivalences

Table 7: Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Table 8: Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

Example 5 (6 in book)

Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution: We could use a truth table to show that these compound propositions are equivalent.

So, we will establish this equivalence by developing a series of logical equivalences, using one of the equivalences in Table 6 at a time, starting with $\neg(p \rightarrow q)$ and ending with $p \wedge \neg q$.

We have the following equivalences.

$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$ by Example 3

$\equiv \neg(\neg p) \wedge \neg q$ by the second De Morgan's law

$\equiv p \wedge \neg q$ by the double negation law

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We have the following equivalences.

$$\begin{aligned}
\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{by Example 3} \\
&\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan's law} \\
&\equiv p \wedge \neg q && \text{by the double negation law}
\end{aligned}$$

Example 6 (7 in book)

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: Solution: We will use one of the equivalences in Table 6 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$.

We have the following equivalences.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan's laws}\\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan's laws}\\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law}\\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the distributive laws}\\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \vee p \equiv F\\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative laws}\\
&\equiv \neg p \wedge \neg q && \text{by the identity laws}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Example 6 (7 in book)

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: Solution: We will use one of the equivalences in Table 6 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$.

We have the following equivalences.

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan's laws} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan's laws} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the distributive laws} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \vee p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative laws} \\
&\equiv \neg p \wedge \neg q && \text{by the identity laws}
\end{aligned}$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Example 7 (8 in book)

Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **T**. (Note: This could also be done using a truth table.)

$$
\begin{aligned}
(p \wedge q) \to (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by Example 3} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan's law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative laws for disjunction} \\
&\equiv T \vee T && \text{by Example 1 and commutative laws for disjunction} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$

Example 7 (8 in book)

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **T**. (Note: This could also be done using a truth table.)

$(p \wedge q) \rightarrow (p \vee q)$	$\equiv \neg(p \wedge q) \vee (p \vee q)$	by Example 3
	$\equiv (\neg p \vee \neg q) \vee (p \vee q)$	by the first De Morgan's law
	$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	by the associative and commutative laws for disjunction
	$\equiv T \vee T$	by Example 1 and commutative laws for disjunction
	$\equiv T$	by the domination law

1.3 Predicates and Quantifiers (1.4 in book)

Predicates

Statements involving variables, such as "$x > 3$", "$x = y + 3$", "$x + y = z$",

Example 1

Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

Solution: We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$." Hence, $P(4)$, which is the statement "$4 > 3$" is true. However, $P(2)$, which is the statement "$2 > 3$," is false.

Example 2 (3 in book)

Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Solution: To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "$1 = 2 + 3$," which is false. The statement $Q(3, 0)$ is the proposition "$3 = 0 + 3$," which is true.

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Example 3 (5 in book)

Let $R(x, y, z)$ denote the statement "$x + y = z$", What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$?

Solution: The proposition $R(1, 2, 3)$ is obtained by setting $x = 1, y = 2$, and $z = 3$ in the statement $R(x, y, z)$. We see that $R(1, 2, 3)$ is the statement "$1 + 2 = 3$", which is true. Also note that $R(0, 0, 1)$, which is the statement "$0 + 0 = 1$", is false.

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Quantifiers

DEFINITION 1: THE UNIVERSAL QUANTIFIER

The universal quantification of P(x) is the statement

"P(x) for all values of x in the domain."

The notation $\forall x\ P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier**. We read $\forall x\ P(x)$ as "for all x $P(x)$" or "for every x $P(x)$". An element for which $P(x)$ is false is called a **counterexample** of $\forall x\ P(x)$.

DEFINITION 2: THE EXISTENTIAL QUANTIFIER

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$".

We use the notation $\exists x\ P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

- The statement $\forall x\, P(x)$ is true when $P(x)$ is true for every x and is false when there is an x for which $P(x)$ is false.
- The statement $\exists x\, P(x)$ is true when there is an x for which $P(x)$ is true and is false when $P(x)$ is false for every x.

$$\neg\left(\exists x\, Q(x)\right) \equiv \forall x\, \neg Q(x).$$

$$\neg\left(\forall x\, P(x)\right) \equiv \exists x\, \neg P(x).$$

Example 4 (8 in book)

Let $P(x)$ be the statement "$x + 1 > x$". What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?
Solution: Because $P(x)$ is true for all real numbers x, the quantification $\forall x\, P(x)$ is true.

Example 5 (9 in book)

Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?
Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x\, Q(x)$. Thus $\forall x\, Q(x)$ is false.

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Example 6 (10 in book)

Suppose that $P(x)$ is "$x^2 > 0$." To show that the statement $\forall x\ P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a counterexample because $x^2 = 0$ when $x = 0$, so that x^2 is not greater than 0 when $x = 0$.

Looking for counterexamples to universally quantified statements is an important activity in the study of mathematics, as we will see in subsequent sections.
When all the elements in the domain can be listed–say, $x_1, x_2, ..., x_n$–it follows that the universal quantification $\forall x\ P(x)$ is the same as the conjunction, $P(x_1) \land P(x_2) \land \cdots \land P(x_n)$, because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

Example 7 (11 in book)

What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution: The statement $\forall x\, P(x)$ is the same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, because the domain consists of the integers $1, 2, 3$, and 4. Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x\, P(x)$ is false.

Example 8 (13 in book)

What is the truth value of $\forall x\, (x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution: The universal quantification $\forall x\, (x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $\left(\frac{1}{2}\right)^2 \ngeq \frac{1}{2}$. Note that $x^2 \geq x$ if and only if $x^2 - x = x(x-1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x\, (x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$). However, if the domain consists of the integers, $\forall x\, (x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$.

TABLE 1 De Morgan's Laws for Quantifiers. (2 in book)

Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg(\exists x\, P(x))$	$\forall x\, \neg P(x)$	For every x, $P(x)$ is false.	There is an x for $P(x)$ which is true.
$\neg(\forall x\, P(x))$	$\exists x\, \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Example 9 (21 in book)

What are the negations of the statements $\forall x\, (x^2 > x)$ and $\exists x\, (x^2 = 2)$?

Solution: The negation of $\forall x\, (x^2 > x)$ is the statement $\neg\forall x\, (x^2 > x)$, which is equivalent to $\exists x\, \neg(x^2 > x)$. This can be rewritten as $\exists x\, (x2 \leq x)$.

The negation of $\exists x\, (x^2 = 2)$ is the statement $\neg\exists x\, (x^2 = 2)$, which is equivalent to $\forall x\, \neg(x^2 = 2)$. This can be rewritten as $\forall x\, (x^2 \neq 2)$.

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Example 10 (22 in book)

Show that $\neg\forall x\,(P(x) \rightarrow Q(x))$ and $\exists x\,(P(x) \wedge \neg Q(x))$ are logically equivalent.

Solution: By De Morgans law for universal quantifiers, we know that $\neg\forall x\,(P(x) \rightarrow Q(x))$ and $\exists x\,(\neg(P(x) \rightarrow Q(x)))$ are logically equivalent. By the fifth logical equivalence in Table 7 in Section 1.2 (1.3 in book), we know that $\neg(P(x) \rightarrow Q(x))$ and $P(x) \wedge \neg Q(x)$ are logically equivalent for every x. Because we can substitute one logically equivalent expression for another in a logical equivalence, it follows that $\neg\forall x\,(P(x) \rightarrow Q(x))$ and $\exists x\,(P(x) \wedge \neg Q(x))$ are logically equivalent.

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