MATH107 Vectors and Matrices

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Vector valued functions

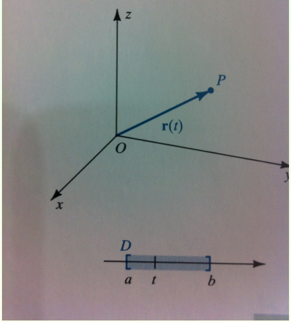
Let D be a set of real numbers $D \in \mathbb{R}$. A vector-valued functions r with domain D is a correspondence that assigns to each number t in D exactly one vector $r(t)$ in V_3 such as

$$r(t) = f(t)i + g(t)j + h(t)k \qquad t \in D$$

where f, g and h are real valued functions called components of vector $r(t)$.

Note: Domain of $r(t)$ is common domain of its components.

Figure 11.1



Examples

Find the domain of $r(t)$

(1) $r(t) = (3+2t)i + \sqrt{1-t}j + t^2k.$

(2) $r(t) = (3+2t)i + (2+t)j + k.$

Examples

Describe the curve defined by the vector valued functions

(1) $r(t) =< 3 + 2t, 1 - t, -2 + 4t >$.

(2) $r(t) =< 2, 4\cos t, 9\sin t >$.

Examples

(1) Let $r(t) = ti + (9 - t^2)j$ for $-3 \leqslant t \leqslant 3$.

a- Sketch the curve C determined by $r(t)$,

b- Sketch $r(t)$ for $t = -3, -2, 0, 2, 3$

(2) Let $r(t) = 3ti + (1 - 9t^2)j$ for $t \in \mathbb{R}$.

a- Sketch $r(0)$ and $r(1)$

b- Sketch the curve C determined by $r(t)$,

Limits

Let $r(t) = f(t)i + g(t)j + h(t)k$. The limit of $r(t)$ as t approaches to a is

$$\lim_{t \to a} r(t) = [\lim_{t \to a} f(t)]i + [\lim_{t \to a} g(t)]j + [\lim_{t \to a} h(t)]k$$

provided f, g and h have limits as t as approaches to a.

Continuity

A vector valued function $r(t)$ is continuous at $t = a$ if

$$\lim_{t \to a} r(t) = r(a).$$

Derivatives

If $r(t) = f(t)i + g(t)j + h(t)k$ and components f, g, and h are differentiable, then

$$\frac{d}{dt} r(t) = \frac{d}{dt} f(t)i + \frac{d}{dt} g(t)j + \frac{d}{dt} h(t)k$$

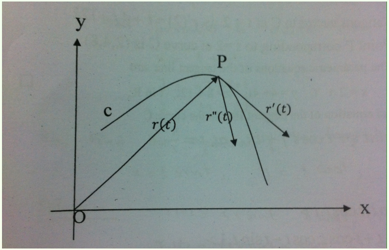
Differentiation Rules

If u and v are differentiable vector-valued functions and c is scalar, then
(1) $\frac{d}{dt}[u(t)+v(t)] = u'(t)+v'(t)$
(2) $\frac{d}{dt}[cu(t)] = cu'(t)$
(3) $\frac{d}{dt}[f(t)u(t)] = f'(t)u(t)+f(t)u'(t)$
(4) $\frac{d}{dt}[u(t).v(t)] = u'(t).v(t)+u(t).v'(t)$
(5) $\frac{d}{dt}[u(t)\times v(t)] = u'(t)\times v(t)+u(t)\times v'(t)$
(6) $\frac{d}{dt}[u(f(t))] = f'(t)u'(f(t))$, Chain Rule

Note 1: The vector $r'(t)$ is called **tangent vector** to the curve at point P.
Note 2: The **tangent line** to the curve C at point P is defined to be line through P and parallel to vector $r'(t)$.
Note 3: The unit tangent vector is

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

Note 4: Geomertical interpretation of $r'(t)$ and $r''(t)$



Note 5: $\lim_{t\to 0} r(t)$ does not exist if one of limit of components $r(t)$ does not exist.

Examples

(1) Find $\lim_{t\to 0} r(t)$, where $r(t) = (1-t)i + 4e^t j + \frac{\sin 2t}{t} k$.
(2) $r(t) = ti + t^2 j + t^3 k, t \geqslant 0$. Find $r'(t)$, $r''(t)$, $r'(t).r''(t)$ and $r'(t) \times r''(t)$. Find the parametric equations of the tangent line when $t = 2$.
(3) Find the parametric equations of the tangent line to c, which given paramerically by $x = 2t^3 - 1, y = -5t^2 + 3, z = 8t + 2$ at point $P(1, -2, 10)$.
(4) $r(t) = ti + 2j + t^2 k$, and $u(t) = i - t^2 j + t^3 k$. Find $\frac{d}{dt}[r(t).u(t)]$ and $\frac{d}{dt}[u(t).u'(t)]$

Definition

Let the position vector for a point $P(x, y)$ $P(x, y, z)$ moving in an xy-plane *solid* be

$$r(t) = x\mathbf{i} + y\mathbf{j} = f(t)\mathbf{i} + g(t)\mathbf{j} \qquad 2D$$

$$r(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad 3D$$

where t is time and f, g and h have first and second derivatives. The velocity, speed and acceleration of P at time t are as follows:

Velocity: $v(t) = r'(t) = \frac{d}{dt}f(t)\mathbf{i} + \frac{d}{dt}g(t)\mathbf{j} + \frac{d}{dt}h(t)\mathbf{k}$

Speed: $\|v(t)\| = \|r'(t)\| = \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$

Acceleration: $a(t) = v'(t) = r''(t) = \frac{d^2}{dt^2}f(t)\mathbf{i} + \frac{d^2}{dt^2}g(t)\mathbf{j} + \frac{d^2}{dt^2}h(t)\mathbf{k}$

Examples

(1) Find velocity, acceleration and speed of $r(t) = t\mathbf{i} + t^3\mathbf{j} + 2t^2\mathbf{k}$ at $t = 1$.

(2) Find velocity, acceleration and speed of $r(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j} + t^2\mathbf{k}$ at $t = \pi/2$.

(3) Find the components of velocity and acceleration at $t = 1$ in direction $b = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, where $x = t^2, y = t - 4, z = t^3 - 3$.

Definition

The indefinite integral of a continuous vector valued function

$$r(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is

$$\int r(t)dt = \Big[\int f(t)dt\Big]\mathbf{i} + \Big[\int g(t)dt\Big]\mathbf{j} + \Big[\int h(t)dt\Big]\mathbf{k}$$

The definite integral of a continuous vector valued function $r(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ on interval $[a, b]$ is

$$\int_a^b r(t)dt = \Big[\int_a^b f(t)dt\Big]\mathbf{i} + \Big[\int_a^b g(t)dt\Big]\mathbf{j} + \Big[\int_a^b h(t)dt\Big]\mathbf{k}$$

Example 1

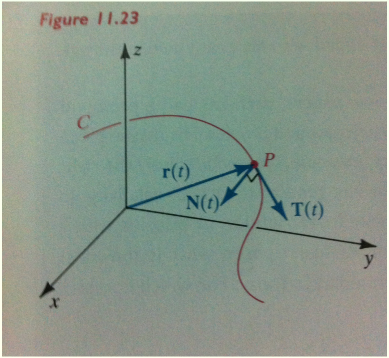
Find the path of the curve when acceleration of the particle moving along this curve is $a(t) = -2\cos t\mathbf{i} - 2\sin t\mathbf{j} + 2\mathbf{k}$, initial velocity of the particle is $v(0) = 2j$ and it starts from point $(2, 0, 0)$.

Unit Tangent Vector

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

Principal Normal Vector

$$N(t) = \frac{T'(t)}{|T'(t)|}$$



Curvature of the curve C when $r(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is

$$\kappa = \frac{|T'(t)|}{|r'(t)|}$$

Curvature of the curve C when $x = f(t), y = g(t)$ is

$$\kappa = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{|(f'(t))^2 + (g'(t))^2|^{\frac{3}{2}}}$$

Curvature of the curve C when $y = f(t)$ is

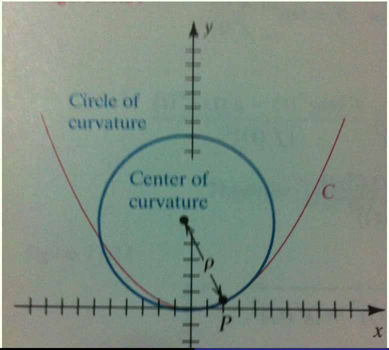
$$\kappa = \frac{|y''|}{|1 + (y')^2|^{\frac{3}{2}}}$$

Radius of Curvature ρ

$$\rho = \frac{1}{\kappa}$$

Centre of Curvature (h, k)

$$h = x - \frac{y'\left(1 + (y')^2\right)}{y''}, \quad k = y + \frac{\left(1 + (y')^2\right)}{y''}$$



Examples

(1) Find unit tangent vector and principal normal vector of the curve $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$.

(2) Find the curvature of the curve given by $r(t) = 2t\mathbf{i} + t^2\mathbf{j} - \frac{1}{3}t^3\mathbf{k}$.

(3) Find the curvature of the curve given by $x = \cos^3 t, y = \sin^3 t$ at point $p(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$.

(4) Find the radius and center of curvature of the curve given by $y = x^4$ at point $P(1, 1)$.

(5) Find the radius and center of curvature of the curve given by $x = t^2, y = t^3$ at $t = 0.5$.

Tangential and Normal components of Acceleration

$$a = a_T T + a_N N$$
$$\|a\|^2 = a_T^2 + a_N^2$$

Tangential Component

$$a_T = \frac{r'(t).r''(t)}{\|r'(t)\|}$$

Normal Component

$$a_N = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|} = \sqrt{\|a\|^2 - a_T^2}$$

Curvature

$$\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = a_N \frac{1}{\|r'(t)\|^2}$$

Examples

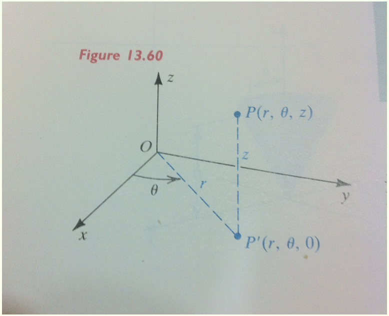
(1) Find unit tangential and normal components of acceleration at time t, when $r(t) = 3t\mathbf{i} + t^3\mathbf{j} + 3t^2\mathbf{k}$. Also find Curvature.

Cylindrical coordinates

Cylindrical coordinates

The cartesian coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point P are related as follows:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \tan\theta = \frac{y}{x},$$
$$r^2 = x^2 + y^2, \quad z = z$$



Spherical coordinates

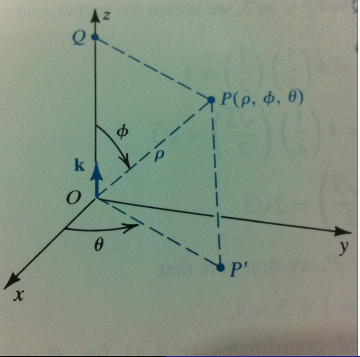
Spherical coordinates

The cartesian coordinates (x, y, z) and the spherical coordinates (ρ, ϕ, θ) of a point P are related as follows:

(1) $x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$

(2) $\rho^2 = x^2 + y^2 + z^2$

Figure 13.71



Spherical to cylindrical

$$r^2 = \rho^2 \sin^2 \phi$$
$$\theta = \theta$$
$$z = \rho \cos \phi$$

cylindrical to Spherical

$$\rho = \sqrt{r^2 + z^2}$$
$$\theta = \theta$$
$$\phi = \left[\frac{z}{\sqrt{r^2 + z^2}} \right]$$

Examples

(1) Express $(x, y, z) = (7, 3, 2)$ in Cylindrical and spherical coordinates.

(2) Find an equation in the spherical coordinates, whose graph is the paraboloid $z = x^2 + y^2$.