PHYS 502

Lecture 10: Wave Laplace and Heat Equations

Solutions of problems in finite domains-c

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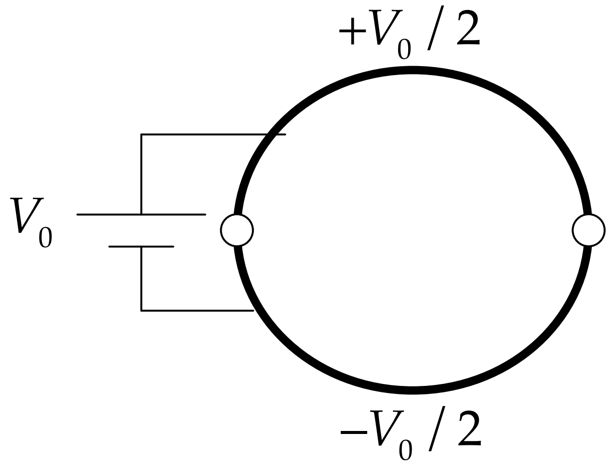
Two dimensional Laplace equation in polar coordinates: The electric field in the interior of a cylindrical capacitor



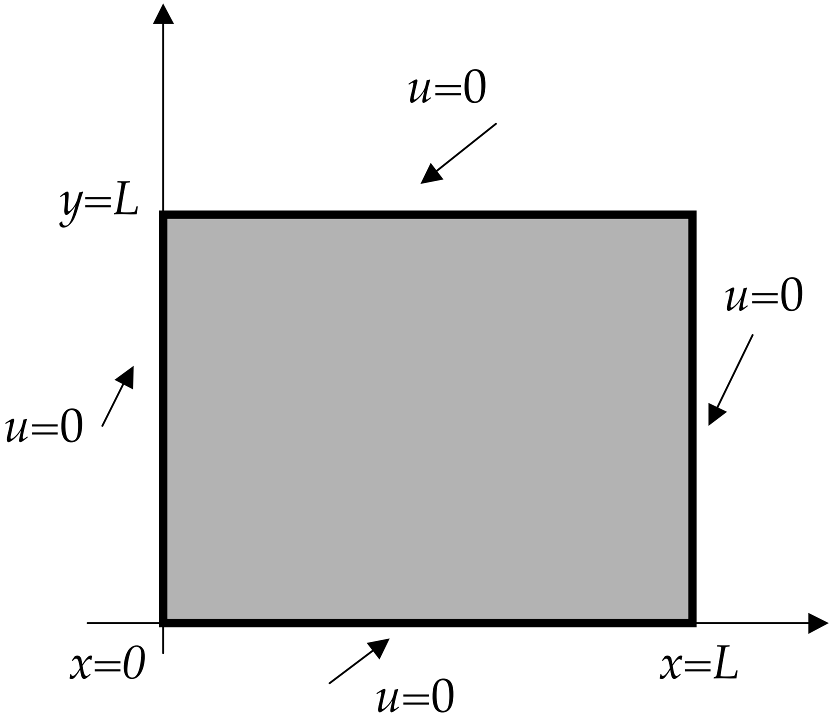
Equation: $\nabla^2 u = 0$

Boundary Conditions: $u(\rho,\theta)\Big|_{\rho=a}^{0<\theta<\pi} = +V_0/2 \qquad u(\rho,\theta)\Big|_{\rho=a}^{\pi<\theta<2\pi} = -V_0/2$

$$u(\rho,\theta) = \frac{2V_0}{\pi}\sum_{n:\,\mathrm{odd}}^{\infty}\frac{1}{n}\left(\frac{\rho}{a}\right)^n \sin n\theta$$

Solution is given in the lecture

Two Dimensional Wave-Equation in Cartesian Coordinates: Vibrations of a square drum



$$\nabla^2 u + k^2 u = 0$$

$$u(0,y,t)=u(L,y,t)=u(x,0,t)=u(x,L,t)=0$$

$$u(x,y,0) = f(x,y), \quad \frac{\partial u}{\partial t}(x,y,0) = g(x,y)$$

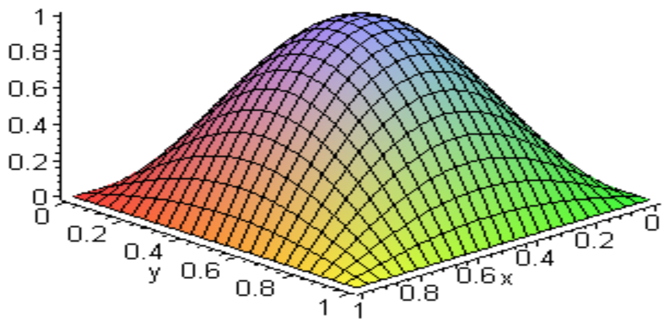
$$u(x,y) = \sum_{n.m} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\left(A_{nm}\cos\omega_{nm}t + B_{nm}\sin\omega_{nm}t\right)$$

$$\omega_{nm} = \frac{c\pi}{L}\sqrt{n^2 + m^2}$$

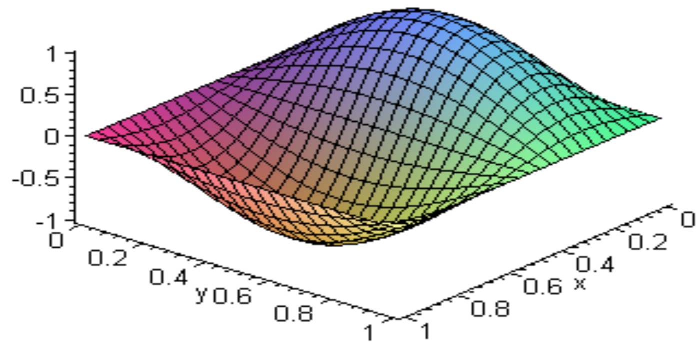
Solution is given in the lecture

Normal modes of a square drum (L=1)

n=1,m=1

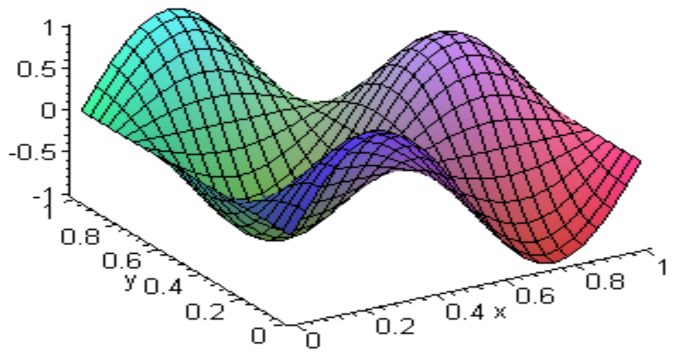


n=2,m=1

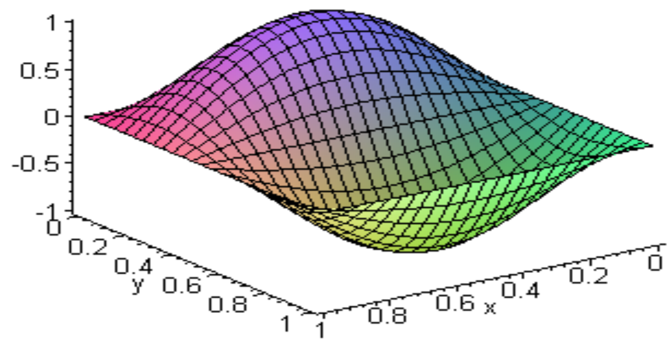


Degeneracy

n=2,m=2

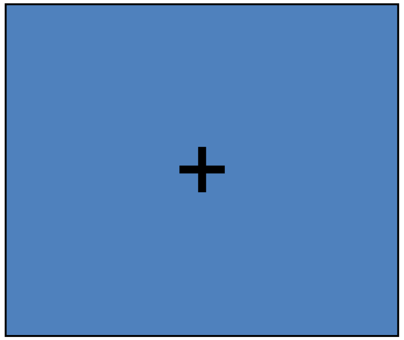


n=1,m=2

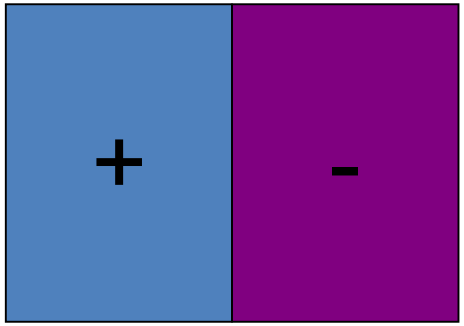


Normal modes of a square drum (L=1)

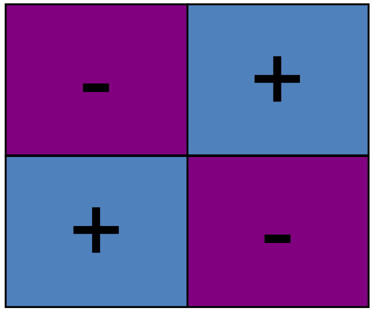
n=1,m=1



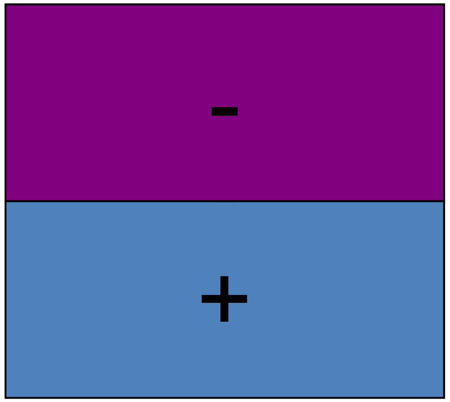
n=2,m=1



n=2,m=2



n=1,m=2

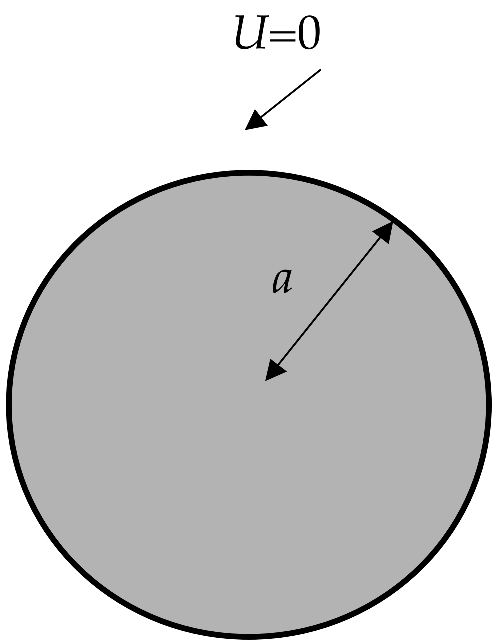


The modes with n=2, m=1 and n=1, m=2 have the same frequency: This effect is called degeneracy

Normal modes of a square drum

- It is obvious that the eigenfrequencies are not multiples of a fundamental frequency (harmonicity) as in the case of an one-dimensional string:
- **The harmonic law does not hold for vibrations in two and/or three dimensions!**

Two Dimensional Wave-Equation in Polar Coordinates: Vibrations of a circular drum-a



$$\nabla^2 u + k^2 u = 0$$

$$u(\rho,\theta)\big|_{\rho=a} = u(a,\theta) = 0$$

$$u(\rho,\theta,0) = f(\rho,\theta)$$

$$u(\rho,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_{nm}\left(x_{nm}\rho / a\right)\left\{A_{nm}\cos n\theta + B_{nm}\sin n\theta\right\}\cos\left(\omega_{nm}t\right)$$

Solution is given in the lecture

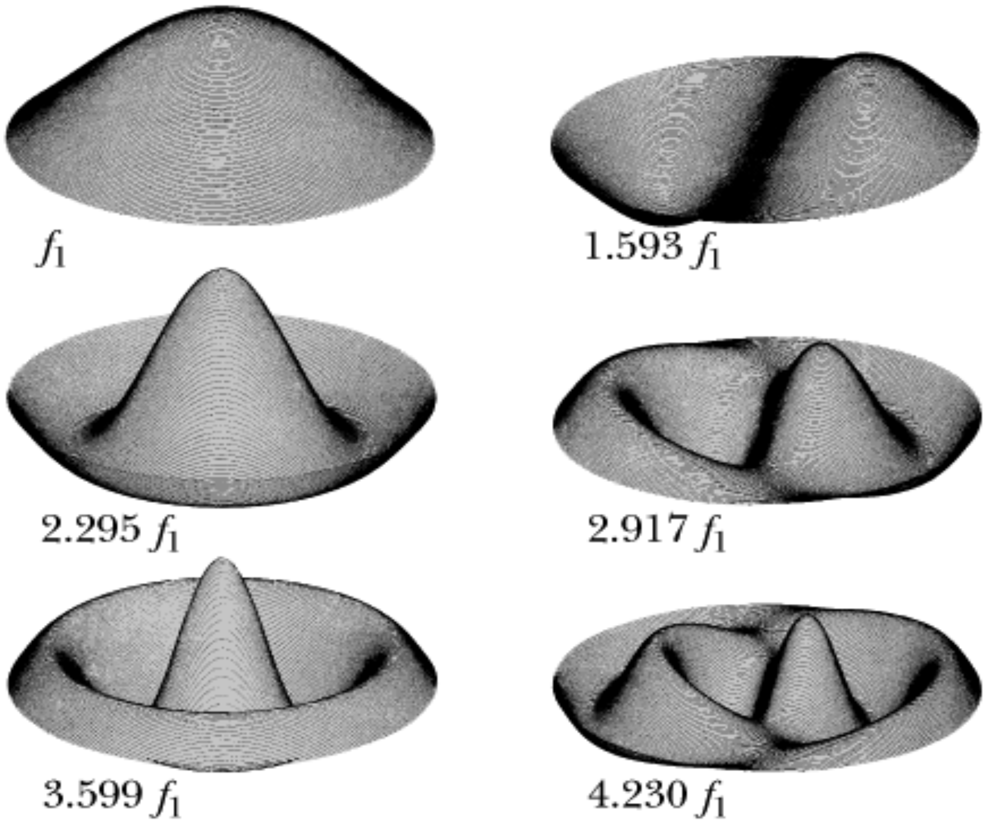
Two Dimensional Wave-Equation in Polar Coordinates: Vibrations of a circular drum-b

$$A_{nm} = \begin{cases} \dfrac{2}{\pi\left[J_{n+1}\left(x_{nm}\rho/a\right)\right]^2}\displaystyle\int_0^1\int_0^{2\pi} \rho f\left(\rho,\theta\right) J_n\left(x_{nm}\rho/a\right)\cos\left(n\theta\right) d\rho d\theta, & n = 1,\ 2,\ 3,\ \ldots \\ \dfrac{1}{\pi\left[J_1\left(x_{0m}\rho/a\right)\right]^2}\displaystyle\int_0^1\int_0^{2\pi} \rho f\left(\rho,\theta\right) J_0\left(x_{0m}\rho/a\right) d\rho d\theta, & n = 0 \end{cases}$$

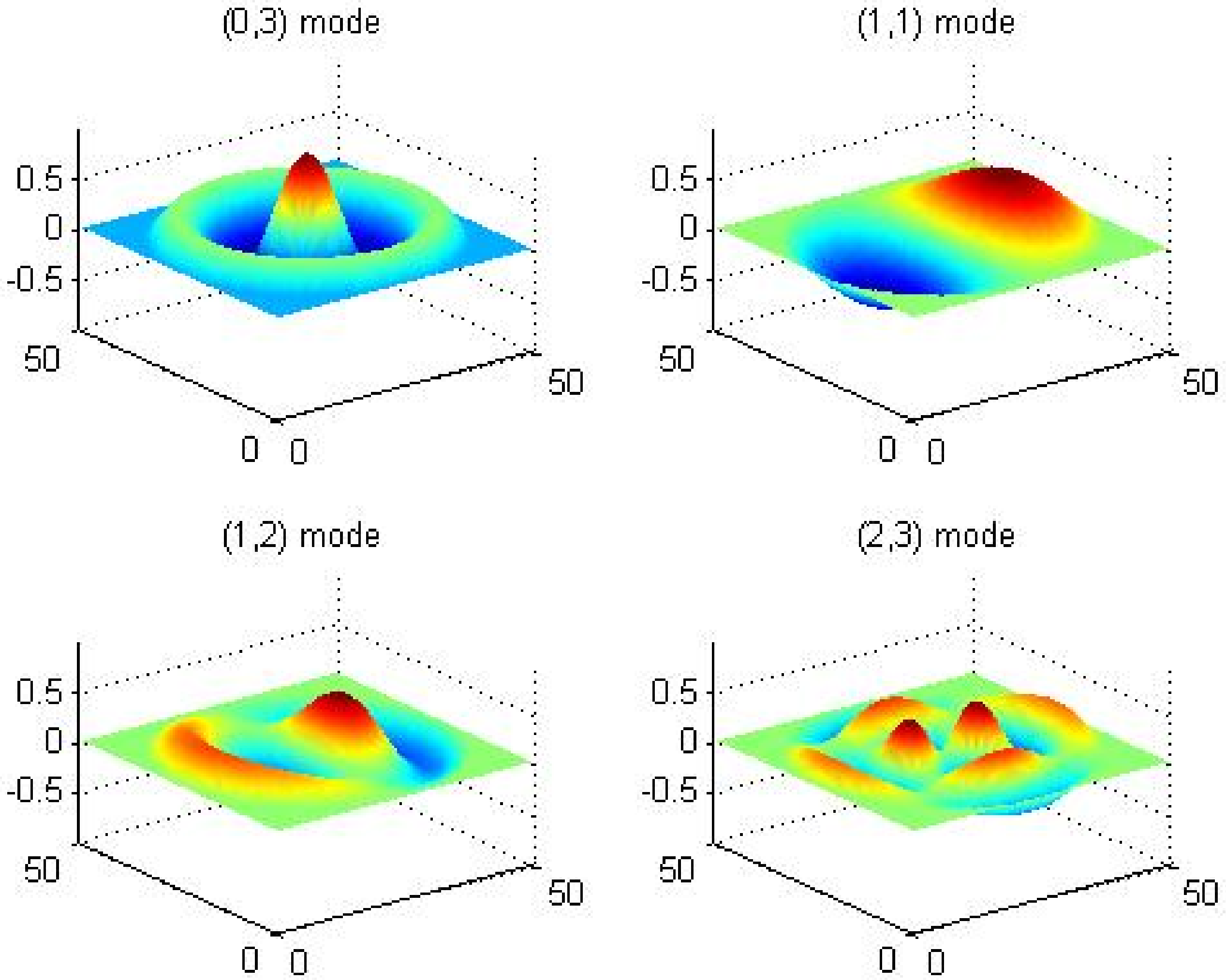
$$B_{nm} = \frac{2}{\pi\left[J_{n+1}\left(x_{nm}\rho/a\right)\right]^2}\int_0^1\int_0^{2\pi} \rho f\left(\rho,\theta\right) J_n\left(x_{nm}\rho/a\right)\sin\left(n\theta\right) d\rho d\theta, \quad n = 0,\ 1,\ 2,\ \ldots$$

Normal modes of a circular membrane-a

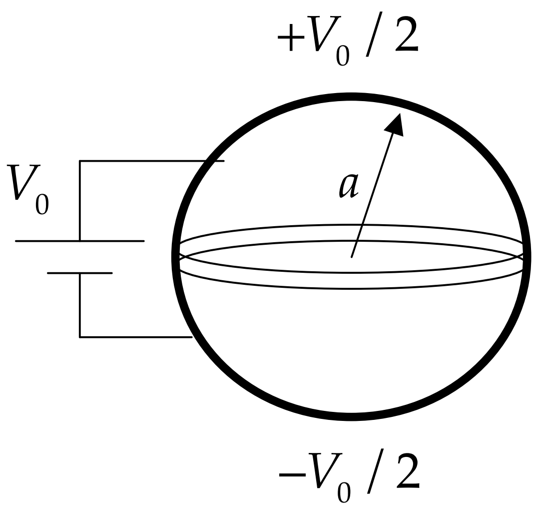
Serway, Physics for Scientists and Engineers, 5/e
Figure 18.17



Normal modes of a circular membrane-b



Three Dimensional Laplace Equation in spherical coordinates: The electric field in the interior of a spherical capacitor



$$\nabla^2 u(r,\theta) = 0$$

$$u(a,\theta) = f(\theta)$$

$$u(r,\theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta)$$

$$c_n = \frac{2n+1}{2a^n} \int_{-1}^{1} P_n(\xi) f(\xi) d\xi$$

Solution is given in the lecture