QUANTUM MECHANICS: LECTURE 9

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Abstract

This lecture discusses the quantum harmonic oscillator by the Ladder operator method

CONTENTS

QUANTIZATION OF THE SHO HAMILTONIAN

From lecture (1) we have the classical Hamiltonian for the simple harmonic oscillator (SHO) :

$$H(p,x) = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

Using the postulates of quantum mechanics discussed before, we obtain-upon quantization - the Hamiltonian operator :

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2 \tag{2}$$

with:

$$[\hat{X},\hat{P}] = i\hbar I \tag{3}$$

The Hilbert space of which $\hat{X}$ and $\hat{P}$ act on is

$$\mathcal{H}(0,+\infty;dx)$$

We now introduce the dimensionless Hamiltonian :

$$\hat{H}' = \frac{1}{2m\hbar\omega}\hat{P}^2 + \frac{1}{2}\frac{m\omega}{\hbar}\hat{X}^2 \tag{4}$$

This operator can be factorised and written in terms of 'creation' and 'inhalation 'operators; $a^\dagger$ and a respectively :

$$\hat{H}' = a^\dagger a + \frac{1}{2}I \tag{5}$$

with:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\hat{X} + i\sqrt{\frac{1}{2m\omega\hbar}}\hat{P}^2 \tag{6a}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{X} - i\sqrt{\frac{1}{2m\omega\hbar}}\hat{P}^2 \tag{6b}$$

These operators, along with $\hat{H}'$, satisfy a well-known commutation relations.

The operators a, $a^\dagger$ and H' along with the commutator operation $[\cdot,\cdot]$ satisfy the $su(1,1)$ algebra.

$$[a,a^\dagger] = I \tag{7a}$$

$$[a,H'] = a \tag{7b}$$

$$[a^\dagger,H'] = -a^\dagger \tag{7c}$$

We also define the **number operator** $N \equiv a^\dagger a$ that acts on the eigenstates $|n\rangle$ resulting an eigenvalue of n :

$$N|n\rangle = n|n\rangle$$

as a result we may conclude that :

$$a|0\rangle = 0 \tag{8}$$

acting on the 'ground state'by the inhalation operator, kills it . Moreover :

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{9}$$

$$a^\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle \tag{10}$$

Hence, The Hamiltonian acting on these states will result (the energy eigenvalue) :

$$\hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle \tag{11}$$

Implying that the 'number states' are the excitation states for the quantum harmonic oscillator. The creation and inhalation operators excite or deceit it, and it has a descrete energy spectrum of :

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) \tag{12}$$

Even in the ground state, the quantum harmonic oscillator has a non-vanishing energy. This is a direct result for the uncertainty principle in time and energy.

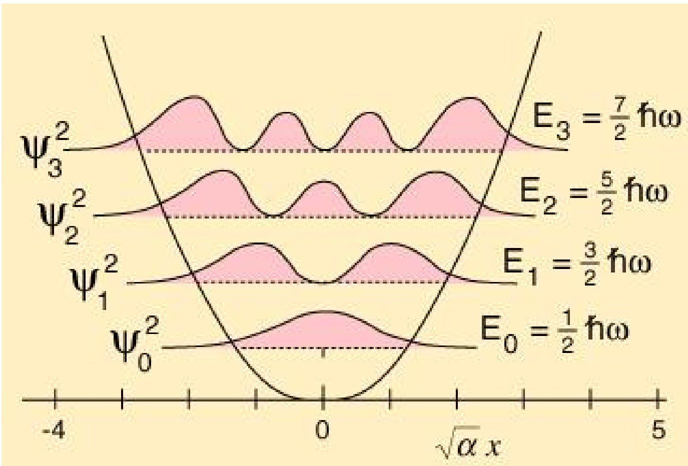

Figure 1: Energy-levels and wavefunctions of the quantum harmonic oscillator

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