PHYS 502

Lecture 9: Wave Laplace and Heat Equations

Solutions of problems in finite domains-b

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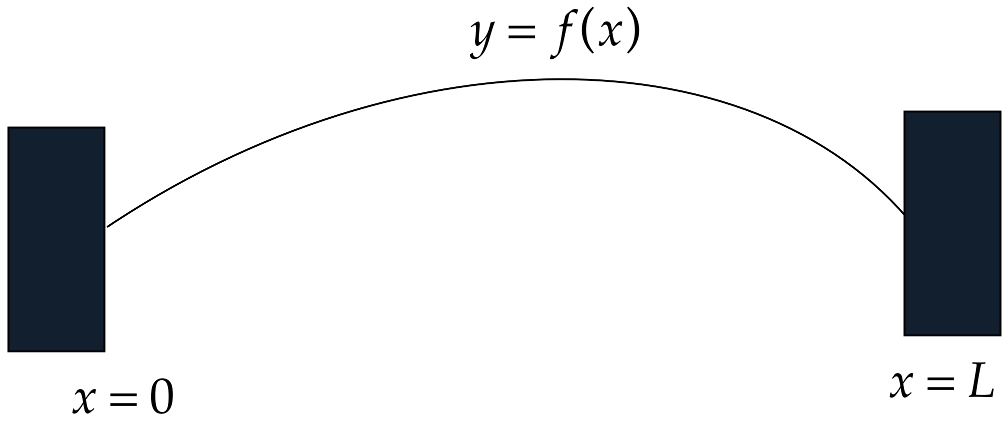
Wave equation in one dimension: A vibrating string



Initial Condition: $u(x,0) = f(x)$

Boundary Conditions: $u(0,t) = u(L,t) = 0 \qquad u_t(x,0) = \partial u(x,t)\partial t\big|_{t=0} = g(x)$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos(n\omega t) + b_n \sin(n\omega t)\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{2}{Ln\omega}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Solution is given in the lecture

Heat equation in one dimension I

Cooling of an infinite slab in a heat bath

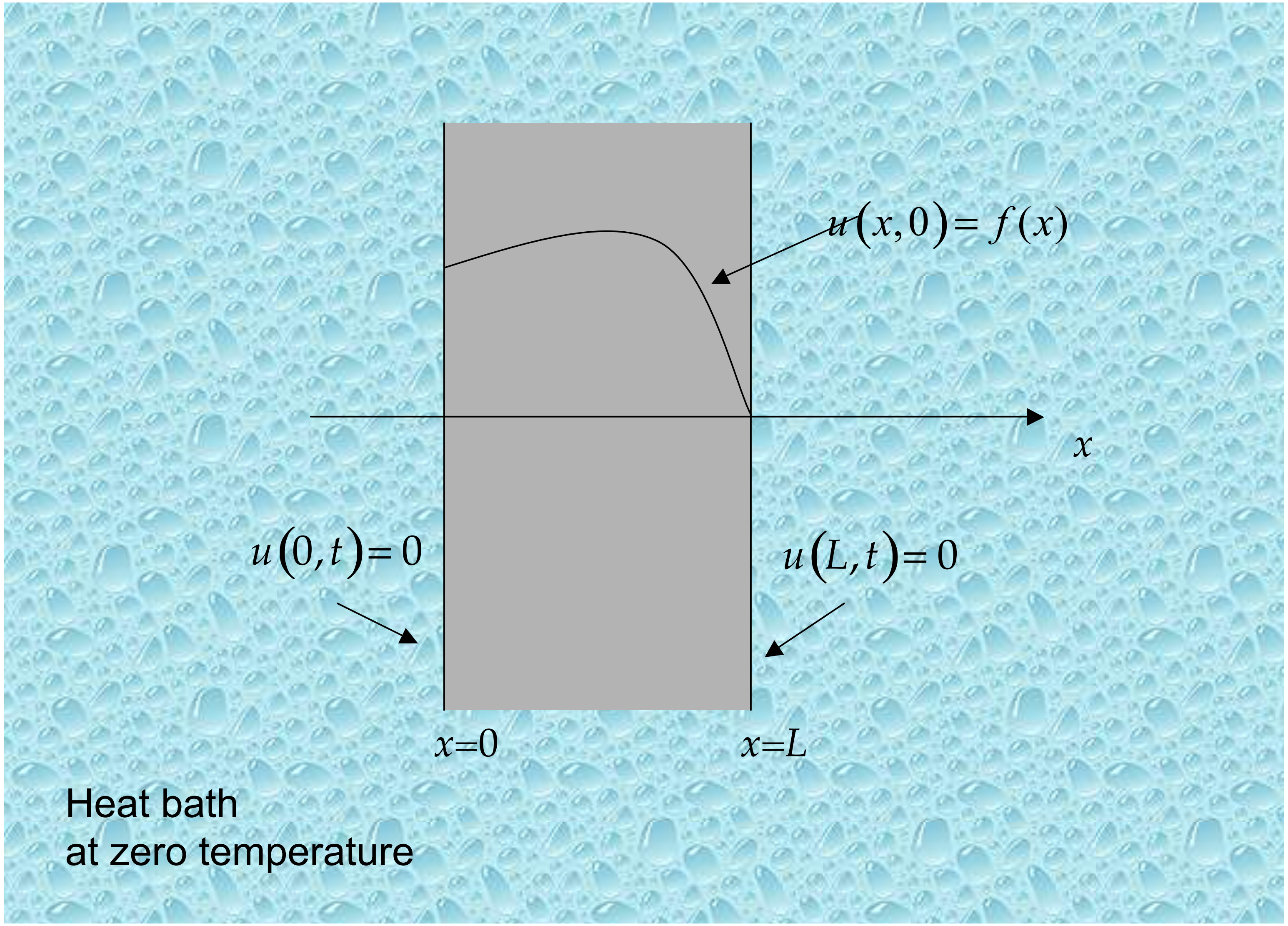
- The problems we are going to consider are characterized by a discrete spectrum.
- A practical problem: A large metallic slab is heated and then we immerse it in a water bath to cool it.

The equation:	$u_t = \sigma u_{xx}$
The boundary conditions:	$u(0,t) = u(L,t) = 0$
The initial condition:	$u(x,0) = f(x)$

$$\text{For } f(x) = T_0, \quad u(x,t) = \frac{4T_0}{\pi} \sum_{n,\ \text{odd}} \frac{1}{n} e^{-n^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

Solution is given in the lecture

Conditions in a slab cooled by a heat bath



Heat equation in one dimension II

*Cooling of an infinite **insulated** slab in a heat bath*

The insulated slab changes the type of the boundary conditions since no heat is transmitted through the boundaries.

The equation:	$u_t = \sigma u_{xx}$
The boundary conditions:	$u_x(0,t) = u_x(L,t) = 0$
The initial condition:	$u(x,0) = f(x)$

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-n^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$$

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad c_0 = \frac{1}{L}\int_0^L f(x)\cos dx$$

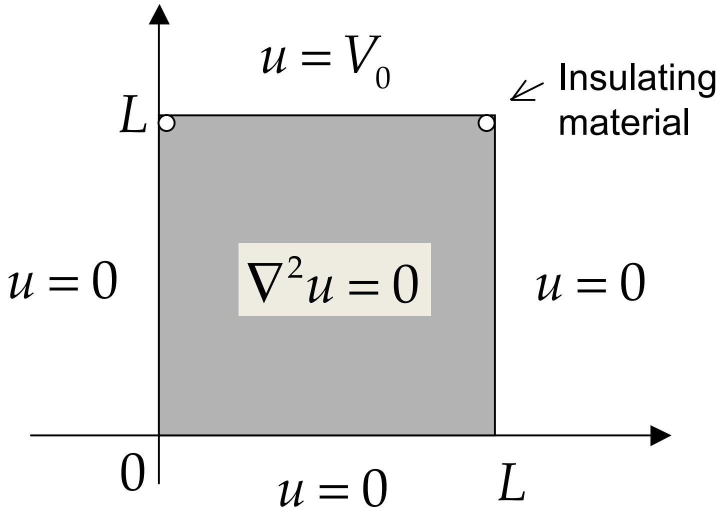
Solution is given in the lecture

Discussion

- The solutions which we got are linear combinations of eigenfunctions each of which has its own rate λ_n. What is the physical content of these rates?
- They describe the relaxation “rates” of the slab towards thermal equilibrium.
- These rates are higher as n gets larger. This is quite natural: Thermal equibrium is achieved by heat transportation from hotter to colder regions. This transportation is proportional to the gradient of the temperature field. The more nodes the more “upside downs” and thus faster relaxation.
- In analogy to normal vibration modes of the wave equation we call them “normal relaxation modes”.

The two dimensional Laplace eq. in cartesian coordinates-a

(Electric field in the interior of a square)



$$\nabla^2 u = 0$$

$$u(0, y) = u(L, y) = 0 \qquad (x - \text{direction})$$

$$\left.\begin{array}{ll} u(x,0) = 0 & \text{homogeneous} \\ u(x,L) = V_0 & \text{non-homogeneous} \end{array}\right\} \qquad (y - \text{direction})$$

The two dimensional Laplace eq. in cartesian coordinates-b

(Electric field in the interior of a square)

$$u(x,y)=\frac{4V_0}{\pi}\sum_{n=\text{odd}}\frac{1}{n\sinh n\pi}\sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$

Solution is given in the lecture