PHYS 502

Lecture 8: Wave, Laplace and Heat Equations

Problems in finite domains-a
Existence and uniqueness of solutions
The method of separating variables

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Introduction-a

- **First.** The variety of partial differential equations that appear in the physical problems is not infinite. On the contrary is very limited. The majority of physical problems could be described (either exactly or approximately) by the following **three** partial differential equations:

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad \text{Wave equation}$$

$$\nabla^2 u = 0 \qquad \text{Laplace Equation}$$

$$\frac{\partial u}{\partial t} - \sigma \nabla^2 u = 0 \qquad \text{Heat Equation}$$

Introduction-b

Second. For all the above three equations the exact solution method (whenever this is possible) is the method of separating variables.

The physical origin of the heat equation-a

It can be shown that the heat equation is the result of two physical laws:

a) The heat equation: $\Delta Q = mc\Delta T$

m: the mass of a body c: the specific heat capacity

ΔT: the change in the temperature

ΔQ: the heat added to or subtracted from the body

The physical origin of the heat equation-b

b) The so-called Fick's law: $\mathbf{j} = -\kappa \nabla T$

which describes the heat transportation from warmer to colder regions.

j: the heat flux vector (heat per unit area and time)

$\varkappa$: the coefficient of thermal conductivity

$T(\mathbf{r}, t)$: the temperature "field" inside the body

$$u_t - \sigma u_{xx} = 0$$

Proof is given in the lecture

The physical origin of the wave equation

- The wave equation is a result of the application of second Newton's law to a segment of a vibrating string.

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0$$

Proof is given in the lecture. Wave equation is a hyperbolic PDE.

The physical origin of the Laplace's equation

a) If we study problems of temperature distribution then after a sufficiently long time the system will be in a temperature equilibrium. In this case the heat equation is reduced into a Laplace equation.

b) When we study problems in electrostatics Laplace equation is a result of the application of Gauss's law in combination with Gauss's theorem:

$$\nabla^2 u = 0$$

Proof is given in the lecture. Laplace equation is an elliptic PDE.

The physical origin of the Heat equation-II
Diffusion effects

- Diffusion effects may be described by the heat equation if we combine Fick's law with Gauss's theorem.
- The fact that the equations which describe the diffusion effects and the heat transportation are identical is not incidental. Heat is transported from regions of high temperatures to regions of lower temperatures. Similarly the diffused substance is transported from regions of high concentration to regions of lower concentrations.
- Moreover in both cases the transports are subject to a common conservation law: The amount of heat in the first case and the amount of diffused mass in the second case.

$$D\nabla^2 c = \frac{\partial c}{\partial t}$$

Proof is given in the lecture. Diffusion equation is a parabolic PDE.

Well defined problems:

(*Initial and boundary conditions that lead to a unique solution*)

In PDE the required additional conditions which lead us to a unique solution are of two different types:

(a) **Initial conditions:** These concern the state of a physical system at a given instant.

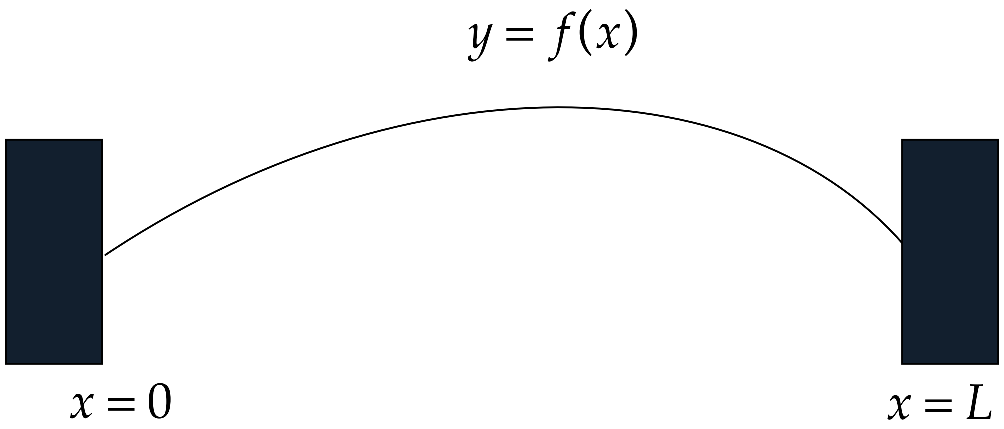
(b) **Boundary conditions:** These concern the values of a physical quantity at the boundary of the physical region of our problem.

Types of boundary conditions

- **Dirichlet:** The value of a function specified on the boundary.
- **Neumann:** The normal derivative (normal gradient) of a function specified on the boundary (e.g. In the electrostatic case this would be the normal component of the electric field and, thus, the surface charge density).
- **Cauchy:** The value of a function and normal derivative specified on the boundary (e.g. In the electrostatic case this would be the potential and the normal component of the electric field).

Type of partial differential equation			
Boundary conditions	**Elliptic** Laplace, Poisson in (x, y)	**Hyperbolic** Wave equation in (x, t)	**Parabolic** Diffusion equation in (x, t)
Cauchy			
Open surface	Unphysical results (instability)	Unique, stable solution	Too restrictive
Closed surface	Too restrictive	Too restrictive	Too restrictive
Dirichlet			
Open surface	Insufficient	Insufficient	Unique, stable solution in one direction
Closed surface	Unique, stable solution	Solution not unique	Too restrictive
Neumann			
Open surface	Insufficient	Insufficient	Unique, stable solution in one direction
Closed surface	Unique, stable solution	Solution not unique	Too restrictive

Boundary conditions in a vibrating string

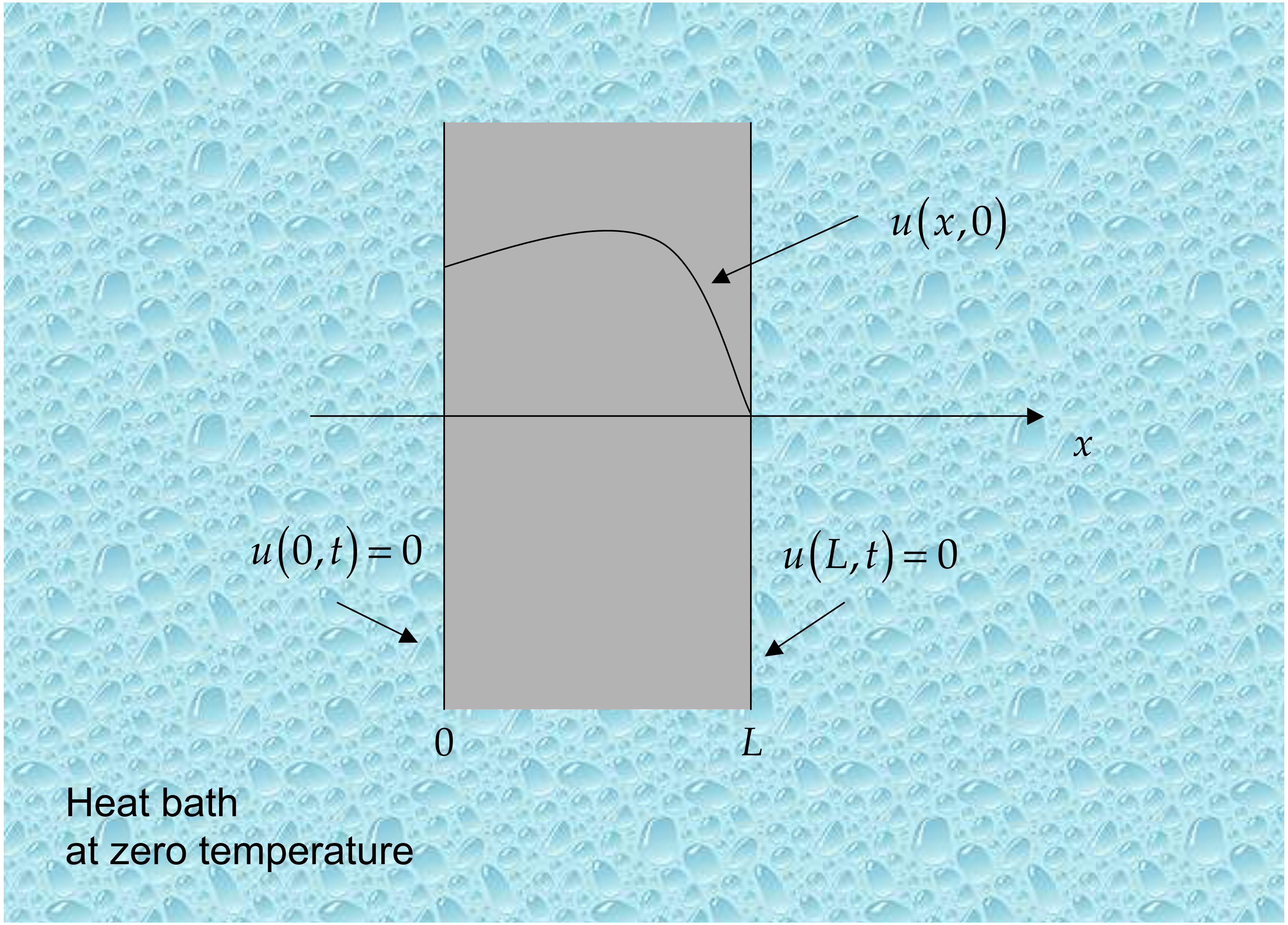


Initial Condition: $u(x,0) = f(x)$

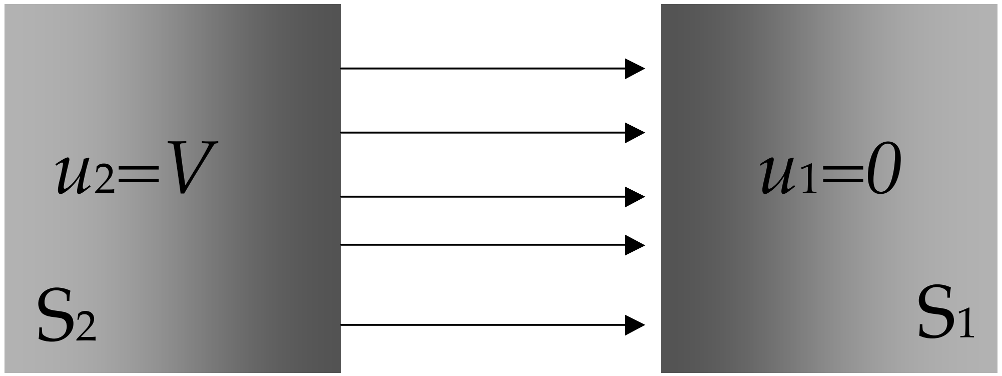
Boundary Conditions: $u(0,t) = u(L,t) = 0$

A string could start a vibration if it is given an initial deformation $u(x,0) = f(x)$
or an initial "kick" at all its points $u_t(x,0) = \partial u(x,t)\partial t\Big|_{t=0} = g(x)$

Conditions in a wall cooled by a heat bath



Boundary conditions in electrostatic problems



Laplace equation does not contain time derivatives. Thus, the necessary conditions are only the boundary ones.

$$u\big|_{S_1} = 0 \qquad\qquad u\big|_{S_2} = V$$

An important remark…

- **When we impose the initial/boundary conditions in the solution of a PDE?**
- Contrary to what is happening in an ordinary differential equation where we first find the general solution and then we impose the initial conditions, in solving a PDE we seek *ab initio the* solution which satisfies the given conditions.
- The reason for this is that the set of solutions of a PDE is huge so either the specific solution is difficult to be found or we cannot apply easily the conditions.

On the validity of the separating variables method -a

- The method of separating variables is very successful for the solution of a PDE. But there is a question: Is it always applicable for the solution of a PDE?
- It can be shown that any PDE of the form

$$\left(L_x + L_y + L_t\right)u\left(x,y,t\right) = 0$$

where L_x, L_y, L_t linear differential operators which act on the corresponding variables. the method of separating variables is *always* applicable.

- We can also prove that the method of separating variables in PDEs of the above form leads to three eigenvalue equations for the relevant operators. The algebraic sum of the eigenvalues is zero.

On the validity of the separating variables method-b

- It can be shown that the method of separating variables is also possible in problems which have symmetries other than cartesian, for example cylindrical or spherical.
- The method of separating variables depends crucially on the type of boundary conditions: *They must be linear and homogeneous*. For example like the ones that follow for a closed interval $[a, b]$:

$$c_1 u_x(a,t) + d_1 u(a,t) = 0$$
$$c_2 u_x(b,t) + d_2 u(b,t) = 0$$

On boundary conditions-a

- By applying the method of separating variables in a PDE we are led in an ODE of the general form: $Ly=\lambda y$, where L is (in the set of problems we are going to deal with) a second order linear differential operator. But there is a question:

> What are the more general linear and homogeneous boundary conditions that can be imposed to a second order DE in order to give us a well defined eigenvalue problem?

On boundary conditions-b

- These conditions are given by:

$$a_{11}y(a)+a_{12}y^{'}(a)+b_{11}y(b)+b_{12}y^{'}(b)=0$$
$$a_{21}y(a)+a_{22}y^{'}(a)+b_{21}y(b)+b_{22}y^{'}(b)=0$$

- The above conditions can be separated in two classes: the *pure* and the *mixed*

On boundary conditions-c

The pure conditions

- Pure conditions are those in which the two limits of interval are not mixed:

$$a_{11}y(a)+a_{12}y^{'}(a)=0 \qquad (b_{11}=b_{12}=0)$$

$$b_{21}y(b)+b_{22}y^{'}(b)=0 \qquad (a_{21}=a_{22}=0)$$

Pure boundary conditions are involved in the majority of physical problems

On boundary conditions-d

The mixed conditions

- On the contrary, from the class of mixed conditions, the only ones that appear in the physical problems are the so called *periodic* boundary conditions:

$$y(a) = y(b), \qquad y^{'}(a) = y^{'}(b)$$

In some problems of theoretical physics we may encounter the so called *anti-periodic* conditions:

$$y(a) = -y(b), \qquad y^{'}(a) = -y^{'}(b)$$

Sturm-Liouiville Theory
Basic points

We present here briefly some points from the so called Sturm-Liouiville theory which describes the problem of eigenvalues.

• The problem of eigenvalues of an operator, in combination with homogeneous boundary conditions has a solution only if the eigenvalue has a discrete set of values (quantization of eigenvalues).

•The set of solutions is called “spectrum”

Sturm-Liouiville Theory

An important theorem-a

- Assume the following problem:

$$Ly = y + \text{ homogeneous boundary conditions}$$

where L is the differential operator

$$L = a(x)d^2 / dx^2 + b(x)d / dx + c(x)$$

with the following preconditions:

a) there is no singular point for the operator L in the closed interval [a, b] at the ends of which we impose the homegeneous boundary conditions of the problem.

b) The homogeneous boundary conditions are of the following two types

I. The function or its derivative is zero at the ends of the interval.

$$a) \quad y(a) = 0, \quad y(b) = 0$$

$$b) \quad y(a) = 0, \quad y'(b) = 0$$

$$c) \quad y'(a) = 0, \quad y(b) = 0$$

$$d) \quad y'(a) = 0, \quad y'(b) = 0$$

Sturm-Liouiville Theory

An important theorem-b

II. The boundary conditions are of periodic character

$$y(a) = y(b), \quad y^{'}(a) = y^{'}(b)$$

Theorem: Any eigenvalue problem with the above conditions has always real eigenvalues and a complete system of orthogonal eigenfunctions.

Sturm-Liouiville Theory

The concept of orthogonality

- Two functions $f(x)$ and $g(x)$ in an interval $[a, b]$, are said to be orthogonal if their *scalar product* is zero, i.e.

$$(f,g)=\int_a^b w(x)f^*(x)g(x)dx=0$$

- Where $w(x)$ is a "weight function" given by

$$w(x)=\pm\frac{1}{a(x)}e^{\int(b(x)/a(x))dx}$$

Where the sign of $w(x)$ is chosen such that $w(x)>0$ in the domain of our problem.

Sturm-Liouiville Theory

The linear combination of eigenfunctions

$$f(x)=\sum_{n=0}^{\infty}c_n y_n(x), \qquad c_n=\frac{\left(y_n,f\right)}{\left(y_n,y_n\right)}$$