Lecture 2

tangent space, differential forms, Riemannian manifolds

differentiable manifolds

A manifold is a set that locally look like $\mathbf{R}^n$. For example, a two-dimensional sphere S^2 can be covered by two subspaces, one can be the northen hemisphere extended slightly below the equator and another can be the southern hemisphere extended slightly above the equator. Each patch can be mapped smoothly into an open set of $\mathbf{R}^2$.

In general, a manifold M consists of a family of open sets U_i which covers M, *i.e.* $\cup_i U_i = M$, and, for each U_i, there is a continuous invertible map $\varphi_i : U_i \to \mathbf{R}^n$. To be precise, to define what we mean by a continuous map, we has to define M as a topological space first. This requires a certain set of properties for open sets of M. We will discuss this in a couple of weeks. For now, we assume we know what continuous maps mean for M. If you need to know now, look at one of the standard textbooks (e.g., Nakahara).

Each (U_i, φ_i) is called a coordinate chart. Their collection $\{(U_i, \varphi_i)\}$ is called an atlas.

The map has to be one-to-one, so that there is an inverse map from the image $\varphi_i(U_i)$ to U_i. If U_i and U_j intersects, we can define a map $\varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j))$ to $\varphi_i(U_i \cap U_j)$. Since $\varphi_j(U_i \cap U_j))$ to $\varphi_i(U_i \cap U_j)$ are both subspaces of $\mathbf{R}^n$, we express the map in terms of n functions and ask if they are differentiable. If the map is differentiable for every intersecting pair of coordinate charts, namely if every change of coordinates is differentiable, then we call M a differentiable manifold.

An important point of this definition of differential manifolds is the following. Suppose there is a function $f : M \to \mathbf{R}$. Consider its restriction on $U_i \cap U_j$. We can express the function in two different sets of coordiantes, $f_i = f \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \mathbf{R}$ and $f_j = f \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \mathbf{R}$. If f_i is differentiable, f_j is also differentiable, and vice versa, since $\varphi_i \circ \varphi_j^{-1}$ and its inverse are both differnetiable. Thus, on a differentiable manifold, we can tell whether a given function is differentiable or not.

We can also give an invariant meaning to differentials of a function as follows.

tangent vectors

A function $f : M \to \mathbf{R}$ is called differentiable if $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbf{R}$ is differentiable for every U_i. Let us denote the space of such differentiable functions by $C^0(M)$.

A tangent vector field v at M is defined as a linear map $C^0(M) \to C^0(M)$ obeying the rule,

$$v(fg) = fv(g) + gv(f).$$

Namely, v behaves like a differential operator on $C^0(M)$.

Note that, when u and v are tangent vector fields, $f \to u(v(f))$ does not give a tangent vector field. On the other hand, their commutator $[u, v] : f \to u(v(f)) - v(u(f))$ is a tangent vector field.

A tangent vector v_p at a point $p \in M$ is a linear map $C^0(M) \to \mathbf{R}$ obeying

$$v_p(fg) = f(p)v_p(g) + g(p)v_p(f).$$

A set of tangent vectors at p is called a tangent space and is denoted by T_pM.

There is another way to think about tangent vectors. Consider two diffentiable curves $c_1, c_2 : \mathbf{R} \to M$. We say that the two curves are tangent at $t = 0$ if $c_1(t = 0) = c_2(t = 0) = p$ and

$$\frac{d}{dt}\varphi(c_1(t))_{|t=0} = \frac{d}{dt}\varphi(c_2(t))_{|t=0}$$

for some coordinate chart containing p. For each curve $c(t)$ with $c(t = 0) = p$, we can define a tangent vector v_p at p by

$$v_p(f) = \frac{d}{dt}f(c(t))_{|t=0}.$$

If c_1 and c_2 are tangent at p, they define the same tangent vector at p. Conversely, any tangent vector can be constructed in this way.

Let us try to express tangent vectors using coordinates. Consider a chart (U, φ), so that $q \in M$ is mapped to $\varphi(q) = (x^1, ..., x^n) \in \mathbf{R}^n$. We can then define a curve $c(t) = \varphi^{-1}(x^1, ..., x^{i-1}, t, x^{i+1}, ..., x^n)$ for fixed x's and a tangent vector e_i at $\varphi^{-1}(x)$ by

$$e_i(f) = \frac{d}{dt}f(c(t))_{|t=x^i} = \frac{\partial}{\partial x^i}f(\varphi^{-1}(x^1, ..., x^n)),$$

or $e_i = \partial_i$ for short. Any tangent vector v_p at $p = \varphi^{-1}(x^1, ..., x^n)$ can then be expressed as

$$v_p = v(p)^i \left(\frac{\partial}{\partial x^i}\right)_p.$$

When v_p is defined in terms of a curve $c(t)$, its component $v(p)^i$ can be obtained by

$$v(p)^i = \frac{d}{dt}\varphi(c(t))^i_{|t=0}.$$

differential forms

Remember what we did in Lecture 1. For each vector space V, we can consider its dual space V^* and their wedge product $\wedge^k V^*$ to define a space of k-forms. We can apply this to the case when $V = T_pM$. Its dual space is $V^* = T^*_pM$ and is called the space of co-tangent vectors.

For a given coordinate chart, a natual basis of T_pM is $e_i = \partial/\partial x^i$. Its dual basis is denoted by $e^i = dx^i$, so that

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j.$$

We can express a k-form ω at $p \in M$ as

$$\omega(p) = \frac{1}{k!}\omega_{i_1 \cdots i_k}(p) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

If the coefficients $\omega_{i_1 \cdots i_k}(p)$ are differentiable functions of p, we call ω as differentiable. Note that this definition of differentiability is independent of a choice of a coordinate chart. The space of differentiable k-forms is denoted by $C^k(M)$ or $\wedge^k T^*M$.

exterior derivative d

The exterior derivative operator d is a map from $C^k(M)$ to $C^{k+1}(M)$. When acting on $f \in C^0$,

$$df(p) = \frac{\partial f}{\partial x^i}(p) dx^i.$$

This is a good notation since d of $x^i : p \to x^i(p)$ gives dx^i. For other forms, d is defined by the requirements,

$$(1)\ d^2 = 0, \quad (2)\ d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^\alpha \alpha \wedge \beta.$$

Here $(-1)^\alpha$ is equal to $+1$ or -1 depending on whether α is an even or odd form.

Question 1: Usin the above definition, show that the exterior derivative of a k-form ω can be expressed in terms of components as

$$d\omega = \frac{1}{k!} \partial_j \omega_{i_1 \cdots i_k} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

So, we can write,

$$d\omega = dx^i \wedge \left(\frac{\partial}{\partial x^i} \omega \right).$$

metric

A metric on M is an element of $T_p^* M \otimes T_p^* M$ at each $p \in M$. It is symmetric and non-degenerate, as the metric on $V \otimes V$ discussed in Lecture 1. It components are given by $g_{ij} = g(\partial_i, \partial_j)$. If g_{ij} is positive definite, (M, g) is called a Riemannian manifold. We can also write this as,

$$ds^2 = g_{ij} dx^i \otimes dx^j.$$

vielbeins, volume form, Hodge * operator

For simplicity, we will assume that the metric g_{ij} is positive definite. For a metric with more general signature, we just have to introduce appropriate sign factors to some of the formulae below.

Since the metric g_{ij} is symmetric, we can find a basis $\{e_i^a\}_{a=1,\cdots,n}$ so that

$$g_{ij} = \sum_{a=1}^{n} e_i^a e_j^a.$$

In the last lecture, we used the symbol $\{e^a\}$ to denote general basis (and used $\mathcal{O}^a$ for orthonormal basis). From now on, we reserve this symbol for the orthonormal frame. For a given metric, the frame is defined modulo $O(n)$.

This can be done at each point p on M. e^a's are called vielbeins. *viel* means many in German, and *bein* is a leg. (In 4 dimensions, they are also called vierbeins or tetrads. In dimensions other than 4, words like fünfbein, etc. have been used. Vielbein covers all dimensions.)

Using the vierbeins, the volume form vol is defined by

$$\text{vol} = e^1 \wedge e^2 \wedge \cdots \wedge e^n.$$

Note that it may not be possible to define vol globally on M since it is invariant under $SO(n)$ but not under $O(n)$. It may not be possible to choose a sign factor for vol (associated to

$Z_2 = O(n)/SO(n)$) consistently over M. The volume form is well-defined if and only if M is orientable.

Using coordinates, we can express the volume form as

$$\text{vol} = \sqrt{g} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

where $g = \det g$ (we are assuming that the metric is positive definite). Or,

$$(\text{vol})_{i_1 i_2 \cdots i_n} = \sqrt{g} \epsilon_{i_1 i_2 \cdots i_n}.$$

For a k-form ω, the Hodge $*$ operator is defined as

$$(*\omega)_{i_{k+1} \cdots i_n} = \frac{1}{k!} \frac{\epsilon^{j_1 \cdots j_k j_{k+1} \cdots j_n}}{\sqrt{g}} \omega_{j_1 \cdot j_k} g_{j_{k+1} i_{k+1}} \cdots g_{j_n i_n}.$$

Here I used the totally anti-symmetric tensor $\epsilon_{i_1 \cdots i_n}$ and $\epsilon^{i_1 \cdots i_n}$ normalized as

$$\epsilon_{12 \cdots n} = \epsilon^{12 \cdots n} = 1.$$

Under coordinate transformations, $\epsilon_{i_1 \cdots i_n}$ does not transform as a tensor. However, we can remedy this by multiplying $\sqrt{g}$ to make it into the volume form. The volume form transforms as a tensor if coordinate transformations preserve the orientation. If we change the orientation, we get an extra (-1).

Question 2: Show

$$* * \omega = (-1)^{k(n-k)} \omega.$$

co-differential δ

The co-differential δ on C^k is defined by

$$\delta \omega = (-1)^{nk+n+1} * d * \omega.$$

The sign is chosen so that δ is *hermitian conjugate* to the exterior derivative d, we we will see later. If $\dim = n$ is even,

$$\delta = - * d * .$$

If n is odd,

$$\delta = (-1)^k * d * .$$

We can easily verify the following properties,

$$\delta^2 = 0, \quad * \delta d = d \delta *, \quad d * \delta = \delta * d = 0.$$

We can use d and δ to define the Laplace-Beltrami operator $\Delta : C^k(M) \to C^k(M)$ as

$$\Delta = \delta d + d \delta.$$

Question 3: Show that, for $f \in C^0(M)$, the Laplace-Beltrami operator is

$$\Delta f = - \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f \right),$$

where $g = \det g$.

integral

We can integrate an n-form ω over an oriented n-dimensional manifold M. Since ω is a top form, it has only one component,

$$\omega = a(x)dx^1 \wedge \cdots \wedge dx^n.$$

The integral is defined by

$$\int_M \omega = \int a(x)dx^1 \cdots dx^n.$$

When M is covered by several coordinate charts, we devide M into segments and use the above in each segment.

Using the integral, we can define an inner product between α and $\beta \in C^k(M)$ by

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta.$$

Question 4: Show that it is symmetric, $(\alpha, \beta) = (\beta, \alpha)$.

If the manifold M has a boundary ∂M, and if $\alpha \in C^n(M)$ is of the form $\omega = d\alpha$ for some $\alpha \in C^{n-1}(M)$, the Stokes theorem holds,

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

In particular, if M has no boundary,

$$\int_M d\alpha = 0.$$

Question 5: Suppose that M has no boundary. For $\alpha \in C^{k+1}(M)$ and $\beta \in C^k(M)$, show

$$(\alpha, \delta\beta) = (d\alpha, \beta).$$

(This is the reason for the choice of sign in the definition of δ.)

supersymmetry

Let us try to compare the mathematics we discussed in this lecture with the fermion picture. As in the previous lecture, we identify the fermion creation operator $\bar{\psi}^i$ with the multiplication of the 1-form

$$\bar{\psi}^i \leftrightarrow dx^i \wedge .$$

This maps a k-form to a $(k+1)$-form. The fermion annihilation operator should map a k-form to a $(k-1)$-form. Thus, we define

$$\psi^i \leftrightarrow (-1)^{nk+k+1} * dx^i * .$$

The sign is chosen so that

$$\{\psi^i, \psi^j\} = 0, \quad \{\bar{\psi}^i \bar{\psi}^j\} = 0, \quad \{\psi^i, \bar{\psi}^j\} = g^{ij},$$

and

$$(\psi^i \alpha, \beta) = (\alpha, \bar{\psi}^i \beta).$$

We also identify, according to the quantum mechanics, a bosonic momentum operator p_i with the derivative $\partial/\partial x^i$.

$$p_i \leftrightarrow -i \frac{\partial}{\partial x^i}.$$

We can then write the exterior derivative operator d and the co-differential δ as,

$$d = i\bar{\psi}^i p_i, \quad \delta = -i\psi^i p_i.$$

Thus,

$$[d, x^i] = \bar{\psi}^i, \quad [\delta, x^i] = -\psi^i,$$

and

$$\{d, \bar{\psi}^i\} = 0, \quad \{d, \psi^i\} = ig^{ij} p_j,$$
$$\{\delta, \bar{\psi}^i\} = -ig^{ij} p_j, \quad \{\delta, \bar{\psi}^i\} = 0.$$

Namely, d and δ generate exchanges between the bosonic oprators p_i and the fermionic operators $\psi^i, \bar{\psi}^i$.

We can also think of the Laplace-Beltrami operator $\Delta = \{d, \delta\}$ as the *Hamiltonian.* On a Riemannian manifold, it is interesting to find the spectrum of Δ. Since d and δ commute with Δ, we can think of d and δ as generating some symmetry. In fact, it is supersymmetry since they exchange bosons and fermions.

We can think of $\sum_{k=0}^n C^k(M)$ as the Hilbert space of the quantum system. The innter product is defined by (α^*, β), so we should restrict the space to be those with normalizable differential forms. With respect to this metric, δ is hermitian conjugate of d, and Δ is hermitian.

In a later lecture, we will see that there is a dynamical system associated to M which has supersymmetry and whose quantization gives $\sum_{k=0}^n C^k(M)$ as the Hilbert space.