Lecture 29

Pb 6.1.13 p. 286

Using the differential Eqns

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) \qquad (1)$$

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t), \quad n = 1, 2, \dots \qquad (2)$$

where all birth parameters are the same constant λ,

with initial condition $X(0) = 0$

show that $\quad P_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \dots$

(poisson process)

Ans: Let $X(t)$ represents the size of the population

, $X(0) = 0$ initial condition

الشرط الابتدائي

$$X(0) = 0 \Rightarrow P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

أي أن العملية بدأت من الحالة 0 بنسبة 100%

$$X(0) = 0 \Rightarrow P_0(0) = 1$$

$$, P_n(0) = 0, \quad n = 1, 2, \dots$$

$$(1) \Rightarrow \frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

Seperation of Variables بفصل المتغيرات

$$\frac{dP_0(t)}{P_0(t)} = -\lambda\, dt$$

and by Integration from 0 to t, we get

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$$\int_0^t \frac{dP_0(u)}{P_0(u)} = -\lambda \int_0^t du$$

$$\therefore \left[\ln P_0(u)\right]_0^t = -\lambda t$$

$$\ln P_0(t) - \ln P_0(0) = -\lambda t \quad , \quad P_0(0) = 1$$

$$\therefore \ln P_0(t) - \cancel{\ln 1}^{\,0} = -\lambda t$$

$$\therefore \ln P_0(t) = -\lambda t \Rightarrow \boxed{P_0(t) = e^{-\lambda t}} \quad ③$$

الحل هنا يمثل دالة الاحتمال عند اللحظة t في الحالة 0.

$$② \Rightarrow \frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t) \quad , \quad n = 1, 2, \dots$$

$$\therefore \frac{dP_n(t)}{dt} + \lambda P_n(t) = \lambda P_{n-1}(t)$$

Multiply both sides by $e^{\lambda t}$

$$e^{\lambda t}\left[\frac{dP_n(t)}{dt} + \lambda P_n(t)\right] = \lambda P_{n-1}(t)\, e^{\lambda t}$$

$$\therefore \frac{d}{dt}\left[e^{\lambda t} P_n(t)\right] = \lambda P_{n-1}(t)\, e^{\lambda t}$$

نفصل المتغيرات

$$d\left[e^{\lambda t} P_n(t)\right] = \lambda P_{n-1}(t)\, e^{\lambda t}\, dt$$

$\therefore$ By Integration from 0 to t

$$\int_0^t d\left[e^{\lambda x} P_n(x)\right] = \lambda \int_0^t P_{n-1}(x)\, e^{\lambda x}\, dx$$

$$\left[e^{\lambda x} P_n(x)\right]_0^t = \lambda \int_0^t P_{n-1}(x)\, e^{\lambda x}\, dx$$

$$e^{\lambda t} P_n(t) - \underset{0}{\cancel{P_n(0)}} = \lambda \int_0^t P_{n-1}(x)\, e^{\lambda x}\, dx$$

$$\therefore\ e^{\lambda t} P_n(t) = \lambda \int_0^t P_{n-1}(x)\, e^{\lambda x}\, dx \quad , n = 1, 2, \dots$$

$$\therefore\ P_n(t) = \lambda e^{-\lambda t} \int_0^t P_{n-1}(x)\, e^{\lambda x}\, dx \quad , n = 1, 2, \dots$$

recurrence relation علاقة تكرارية (4)

at $n=1$

(4) $\Rightarrow$ $P_1(t) = \lambda e^{-\lambda t} \int_0^t P_0(x)\, e^{\lambda x}\, dx$

(3) $\Rightarrow$ $P_0(x) = e^{-\lambda x}$

$$\therefore\ P_1(t) = \lambda e^{-\lambda t} \int_0^t \cancel{e^{-\lambda x}}\, \cancel{e^{\lambda x}}\, dx$$

$$P_1(t) = \lambda e^{-\lambda t} \int_0^t dx = \lambda e^{-\lambda t} \left[x\right]_0^t$$

(5)

$$\therefore\ P_1(t) = \lambda t\, e^{-\lambda t}$$

at $n = 2$

(4) $\Rightarrow$ $$P_2(t) = \lambda e^{-\lambda t} \int_0^t P_1(x)\, e^{\lambda x}\, dx$$

(5) $\Rightarrow$ $$P_1(x) = \lambda x\, e^{-\lambda x}$$

$$\therefore\ P_2(t) = \lambda e^{-\lambda t} \int_0^t \lambda x\, \cancel{e^{-\lambda x}}\, \cancel{e^{\lambda x}}\, dx$$

$$P_2(t) = \lambda^2 e^{-\lambda t} \int_0^t x\, dx$$

$$P_2(t) = \lambda^2 e^{-\lambda t} \left[\frac{x^2}{2}\right]_0^t$$

$$P_2(t) = \lambda^2 e^{-\lambda t} \left(\frac{t^2}{2}\right)$$

$$\therefore\ P_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2!} \qquad (6)$$

From (3), (5) and (6) We deduce that

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \dots$$