PHYS 507
Lecture 2: Advanced definitions

ADVANCED DEFINITIONS

- In the preceding sections vectors were defined or represented in two equivalent ways: (1) geometrically by specifying magnitude and direction and (2) algebraically by specifying the components relative to cartesian coordinate axes.
- The definition of a vector as a quantity with magnitude and direction breaks down in advanced work. On the one hand we encounter quantities, such elastic constants and index of refraction in anisotropic materials, that have magnitude and direction **but** are not vectors. On the other hand, our naïve approach is awkward to generalize, to extend to more complex quantities. We seek a new definition of vector field, using our displacement vector **r** as our prototype.

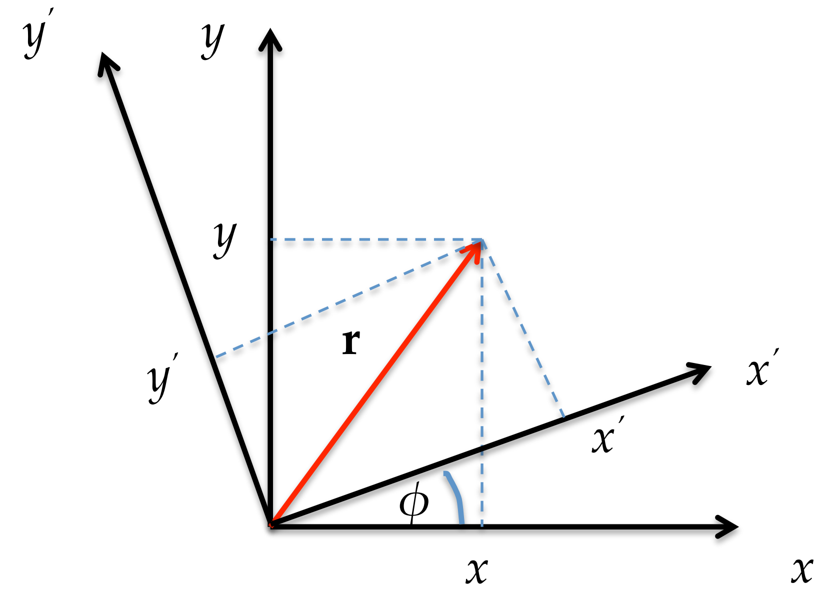
ADVANCED DEFINITIONS

- There is an important physical basis for our development of a new definition: we describe our physical world by mathematics but it and any physical predictions we may make must be **independent** for our mathematical analysis.
- We assume that space is **isotropic**: that is, there is no preferred direction or all directions equivalent. Then the physical system being analyzed or the physical law being enunciated cannot and must no depend on our choice or **orientation** of the coordinate axes.

ADVANCED DEFINITIONS

- We consider first, for simplicity, the two dimensional case and we consider a new system of axes rotated at an angle ϕ. The relations of the new components to the old ones are:

$$x' = x\cos\phi + y\sin\phi$$
$$y' = -x\sin\phi + y\cos\phi$$



ADVANCED DEFINITIONS

- We have seen that a vector could be represented by the coordinates of a point; that is, the coordinates were proportional to the vector components.
- Hence the components of a vector must transform under rotation as coordinates of a point (such as **r**). Therefore any pair of quantities $A_x(x,y)$ and $A_y(x,y)$ in the xy-coordinate system is transformed to (A_x', A_y') by this rotation of the coordinate system with

$$A_x' = A_x \cos\phi + A_y \sin\phi$$

$$A_y' = -A_x \sin\phi + A_y \cos\phi$$

ADVANCED DEFINITIONS

- We define A_x and A_y as the components **A**.
- Our vector now is defined in terms of the transformation of its components under rotation of the coordinate system. If A_x and A_y transform in the same way as x as y, the components of the two-dimensional displacement vector, they are the components of a vector **A**.
- The components of **A** in a particular coordinate system constitute the *representation* of **A** in that coordinate system

EXTENSION TO 3 AND MORE DIMENSIONS

- Consider the following compact notation:

$$x \to x_1, \quad y \to x_2$$

Direction cosines

$$a_{11} = \cos\phi \qquad a_{12} = \sin\phi$$

$$a_{21} = -\sin\phi \quad a_{22} = \cos\phi$$

- Then we have:

$$x_1^{'} = a_{11}x_1 + a_{12}x_2$$
$$x_2^{'} = a_{21}x_1 + a_{22}x_2$$

$$x_i^{'} = \sum_{j=1}^{2} a_{ij}x_j, \quad i = 1,\ 2$$

EXTENSION TO 3 AND MORE DIMENSIONS

- The generalization now to N coordinates is very simple:

$$V_i' = \sum_{j=1}^{N} a_{ij} V_j, \quad i = 1,\ 2,\ \ldots\ , N$$

- As before, a_{ij} is the cosine of the angle between x_i' and x_j.

$$a_{ij} = \frac{\partial x_i'}{\partial x_j} = \frac{\partial x_j}{\partial x_i'}$$

$$V_i' = \sum_{j=1}^{N} \frac{\partial x_i'}{\partial x_j} V_j = \sum_{j=1}^{N} \frac{\partial x_j}{\partial x_i'} V_j$$

$$\sum_i a_{ij} a_{ik} = \delta_{jk}, \quad \sum_i a_{ji} a_{ki} = \delta_{jk}$$

ADVANCED DEFINITIONS

In redefining a vector in terms of how its components transform under a rotation of the coordinate system, we should emphasize two points:

- This definition is developed because it is useful and appropriate in describing our physical world. Our vector equations will be independent of any particular coordinate system. The coordinate system need not even be Cartesian.
- This definition is subject to generalization that will open up the branch of mathematics known as tensor analysis.