Lecture 12

geometry of gauge theory

Gauge Fixing

The Yang-Mills Lagrangian density takes the form,

$$\mathcal{L} = \frac{1}{4}\mathrm{tr}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)^2.$$

We can choose a basis $\{T^a\}$ for the Lie algebra of the gauge group G and write

$$A_\mu = \sum_a A_\mu^a T^a.$$

This action does not contain the time derivative of the variable A_0, which is the source of fun. Let us denote spacelike directions by the indix i. The action can then be expressed as,

$$\mathcal{L} = \mathrm{tr}\left(\frac{1}{2}E_i\dot{A}_i - \frac{1}{4}(E_iE_i + B_iB_i) + \frac{1}{2}A_0 C\right),$$

where

$$E_i = F_{0i}, \quad B_i = \epsilon_{ijk}F_{jk},$$

and

$$C = \partial_i E_i + [A_i, E_i].$$

If we view the Yang-Mills theory as a classical mechanical system and apply the standard Lagrangian formalism, we see that E_i and A_i are canonically conjugate to each otehr with the Poisson bracket,

$$\{E_i^a(t,\vec{x}), A_j^b(t,\vec{y})\} = \delta_{ij}\delta^{ab}\delta(\vec{x}-\vec{y}),$$

where a, b refer to gauge group indices and we assumed that the basis of the Lie algebra is chosen so that $\mathrm{tr}T^aT^b = \delta^{ab}$. The time component A_0 does not have its conjugate variable, and it serves as a Lagrange multiplier to impose the constraint, $C = 0$. We see that the Poisson bracket for E_i and A_i imply

$$\{C(\vec{x})^a, C(\vec{y})^b\} = f^{abc}C^c(\vec{x})\delta(\vec{x}-\vec{y}),$$

where we suppressed the time coordinate t, which are common to all, and f^{abc} is the structure constant defined by $[T^a, T^b] = if^{abc}T^c$. Thus, it is consistent to impose the condition $C = 0$.

It is important to note that

$$\{C_\mu, A_i(\vec{x})\} = D_i\mu = \partial_i\mu + [A_i, \mu],$$

where

$$C_\mu = \int d\vec{y}\ \mathrm{tr}\left(\mu^a(\vec{y})C^a(\vec{y})\right).$$

Namely, $C^a(\vec{x})$ is the generator of the gauge transformation, $A_i \to A_i + D_i\mu$. In particular, the Hamiltonian

$$H = \int d\vec{x}\mathrm{tr}\left(E_iE_i + B_iB_i\right),$$

which is gauge invariant, commutes with C.

If we have constraints that are closed under the Poisson bracket, they are called the first class.

Symplectic Reduction

To understand the phase space of the Yang-Mills theory, it is useful to review the notion of the symplectic reduction. A phase space M is an even dimensional space (say $2m$ dimensional) with a non-degenerate Poisson bracket. Locally, we can choose canonical coordinates (q^i, p_i) $(i = 1, ..., m)$ so that $\{q\ i, p_j\} = \delta^i_j$. Suppose we have a Hamiltonian H and a set of constraints $\{\varphi^a\}$ $(a = 1, ..., n)$, satisfying,

$$\{\varphi^a, \varphi^b\} = \sum_c c^{ab}_{\ \ c}\varphi^c, \quad \{H, \varphi^a\} = \sum_b d^a_b \varphi^b,$$

for some functions $c^{ab}_{\ \ c}, d^a_b$ on M. These are of the first class.

To impose the constraints, the Lagrangian is defined as

$$L = p_i \dot{q}^i - H - \sum_a \lambda_a \varphi^a,$$

where λ_a's are Lagrange multipliers.

Question 1: Derive the equations of motion for (q, p) with keeping the Lagrange multipliers as arbitrary functions of t, and show that trajectories stay within the constrainted subspace in M.

Trajectories depend on λ's, but we would like time evolutions of physical observables be independent of them. Namely, observable functions f should satisfy

$$\{f, \varphi^a\} = 0 \quad (\mathrm{mod}\varphi).$$

This means that we should not only evaluate f on the $(2m - n)$-dimensional subspace with $\varphi^a = 0$, but f should be independent of n more directions generated by φ's.

To paramatrize the additional n directions, let us choose functions $\chi_a(q, p)$ $(a = 1, ..., n)$ so that

$$\det\{\chi_a, \varphi^b\} \neq 0.$$

For simplicity, let us also assume $\{\chi_a, \chi_b\} = 0$, though we can relax this condition. Since φ_a's do not Poisson commute with the constraints φ^a, they are not physical.

Consider a new subspace M^* defined by the two sets of constraints, $\chi_a = 0, \varphi^a = 0$. They do not Poisson commute and they are called the second class constraints.

Since χ_a commute with each other, we can choose canonical coordinates on M such that the first n momenta are $p_a = \chi_a$ $(a = 1, ..., n)$. Their conjugate coordinates are q^a. Let us denote the rest of canonical coordinates by (q^s_*, p_{*s}) $(s = 1, ..., m - n)$. Since χ_a and φ^a have non-degenerate Poisson bracket,

$$\det\left(\frac{\partial \varphi^a}{\partial q^b}\right) = \det\{\varphi^a, \chi_b\} \neq 0.$$

This means that we can solve the constraints $\varphi^a = 0$ on M^* by choosing q^a appropriately. Namely, we can characterize M^* as a subspace of M obeying,

$$p_a = 0, \quad q^a = q^a(q_*, p_*).$$

In particular, (q_*, p_*) are natural canonical coordinates on M^* and we can use them to define the Poisson bracket on $M*$. This procedure, to derive the new phase space $M*$ from the old phase space M subject to the constraints φ^a, is called the symplectic reduction.

This can be done whenever a group G is acting on a phase space G as a canonical transformation. Then there is a generator (or a set of generators) φ for the G action. Define $M_0 = \varphi^{-1}(0)$, namely the subspace of M where $\varphi = 0$. There is a G action on M_0 and that the quotient M_0/G is naturally a phase space (i.e., with non-degenerate symplectic form). This is the same as M^* discussed here. The reduced phase is also denoted as $M//G$. It is also called a symplectic quotient or a Marsden-Weinstein quotient.

Faddeev-Popov Determinant

When one quantize a system using the path integral, the integral should be over trajectories on the physical phase space M^*, so that the resulting quantum amplitudes obey the unitarity conditions. On the other hand, it is often convenient to write the integral using canonical coordinates on M. The Jacobian for the change of variables is called the Faddeev-Popov determinant.

Consider the natural measure ω_* on M^*,

$$\omega_* = \prod_{s=1}^{m-n} dq_*^s dp_{*s},$$

and compare it with

$$\omega = \prod_{i=1}^{m} dq^i dp_i.$$

Since $M*$ is a subspace of M with the conditions, $p_a = 0, q^a = q^a(q_*, p_*)$, the two measures are related as

$$\omega_* = \prod_{a=1}^{n} \delta(q^a = q^a(q_*, p_*))\delta(p_a)\omega_*.$$

Since $p_a = \chi_a$ and $q^a = q^a(q_*, p_*)$ solve $\varphi(q,p) = 0$, we can write,

$$\prod_a \delta(q^a)\delta(p_a) = \prod_a \delta(\chi_a)\delta(\varphi^a)\det\left(\frac{\partial \varphi^a}{\partial q\ b}\right) = \prod_a \delta(\chi_a)\delta(\varphi^a)\det\{\varphi^a, \chi_b\}.$$

If the Lagrange multiplier term $\lambda_a \varphi^a$ is added to the Lagrangian, we can replace the constraint φ^a by an integral over λ_a. Thus, the measure ω_* on the physical phase space can be replaced by,

$$\det\{\varphi^a, \chi_b\} \prod_a \delta(\chi_a) d\lambda_a \prod_i dq^i dp_i.$$

The result of the integral is independent of the choice of χ_a as far as $\det\{\chi_a, \varphi^b\} \neq 0$.

Going back to the Yang-Mills theory, we see that the constraints are $C(\vec{x}) = 0$ and the Lagrange multipliers are $A_0(\vec{x})$. As the second set of constraints χ, we should choose ones that do not Poisson commute with $C(\vec{x})$, namely gauge non-invariant conditions. They are gauge fixing conditions. For example, in the Coulomb gauge, we choose $\partial_i A_i = 0$ as such conditions. In this case, the Faddeev-Popov determinant is $\det \partial_i D_i$.

Geometry of Gauge Field Configurations

So far, we have looked at local geometry of gauge fixing in the phase space picture. Let us turn to global structure of the physical phase space of the Yang-Mills theory. Consider the four-dimensional Yang-Mills theory on S^4. Call the gauge field configuration space as $\mathcal{A}$ and the group consisting of gauge transformations by $\mathcal{G}$, which is a subset of $\Omega^4(G) = \{g(x) : S^4 \to G\}$. The quotient $\mathcal{A}/\mathcal{G}$ is a set of physically inequivalent gauge field configrations. One can show that this is homotope to $\Omega^3(G)$, i.e. space of maps from S^3 to G. This comes from separating S^4 into the northern and southern hemispheres and gluing them by gauge transformations across the equator $\sim S^3$.

Let us study the topology of this infinite dimensional space, $\Omega^3(G)$. To do so, we note that $\Omega^n(\Omega^m(G))$ is homotope to $\Omega^{n+m}(G)$. This in particular means that $\pi_n(\Omega^m(G)) = \pi_{n+m}(G)$. This simplifies our task since we know how to compute the homotopy of the group G itself.

For example, if $G = SU(N)$, $\pi_0(\Omega^3(G)) = \pi_3(G) = \mathbf{Z}$ if $N \geq 2$. This means that the gauge field configuration space $\mathcal{A}/\mathcal{G}$ consists of infinitely many disjoint components parametrized by $\mathbf{Z}$. In fact, this integer parameter is nothing but the second Chern number (instanton number),

$$C_2 = \frac{1}{8\pi^2} \int_{S^4} \mathrm{tr} F \wedge F.$$

The index theorem discussed in Lecture 8 tells you that this number is related to the number of zero modes of the Dirac operator coupled to the gauge field.

We have $\pi_1(\Omega^3(G)) = \pi_4(G) = 0$ for $N \geq 3$, so the space of gauge transformations is simply connected for $SU(N \geq 3)$. However, $\pi_1(\Omega^3(SU(2)) = \pi_4(SU(2)) = \mathbf{Z}_2$. Since $\pi_0(\Omega^4(G))$ is also equal to $\pi_4(G)$, this means that, for $G = SU(2)$, there is topologically nontrivial gauge transformation on S^4 that cannot be continuously deformed to the identity. It was shown by E. Witten that, if there is an odd number of chiral fermions that are in the doublet representation of $SU(2)$, its Dirac determinant changes the sign under the $\mathbf{Z}_2$ action of the gauge symmetry. This makes it impossible to quantize such a model consistently with the gauge symmetry. This is called the Witten anomaly.

For $N \geq 3$, we also have $\pi_2(\Omega^3(G)) = \pi_5(G) = \mathbf{Z}$ (it is $\mathbf{Z}_2$ for $SU(2)$). This is related the so-called non-abelian anomaly, a violation of the gauge symmetry and non-conservation of the Noether current for the gauge symmetry.

Question 2: Repeat this analysis in other spacetime dimensions.