QUANTUM MECHANICS: LECTURE 11

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Abstract

This lecture continues the discussion of the angular momentum in quantum mechanics

CONTENTS

ANGULAR MOMENTUM EIGENSTATES

We now use more abstract method to analyse the angular momentum spectrum, by introducing the eigenstates for L^2 and L_3

$$|\beta, m\rangle \tag{1}$$

such that :

$$L^2|\beta, m\rangle = \beta|\beta, m\rangle \tag{2a}$$
$$L_3|\beta, m\rangle = m\hbar|\beta, m\rangle. \tag{2b}$$

Now we look at the effect of the operators $L_\pm$ on the eigenstates:

$$\begin{aligned} L_3 L_\pm|\beta, m\rangle &= L_\pm L_3|\beta, m\rangle + [L_3, L_\pm]|\beta, m\rangle \\ &= L_\pm(L_3 \pm I)|\beta, m\rangle \\ &= L_\pm(m\hbar \pm 1)\beta, m\rangle \\ \Rightarrow L_\pm|\beta, m\rangle &= |\beta, m \pm 1\rangle \end{aligned} \tag{3}$$

Therefore, the operators $L_\pm$ acting on the eigenstates rise / lower the state, just like the creation and annihilation operators seen in the quantum harmonic oscillator. In fact, the operator L_+ rotates the angular momentum towards the $z$axis , whilst L_- rotates it away from the z axis towards the $-z$ axis.

THE SPECTRUM OF ANGULAR MOMENTUM OBSERVABLE

Now consider the following expected values :

$$\langle L_3^2 \rangle = \hbar^2 m^2 \tag{4a}$$
$$\langle L^2 \rangle = \langle L_1^2 \rangle + \langle L_2^2 \rangle + \langle L_3^2 \rangle$$
$$\beta = a^2 + b^2 + \hbar^2 m^2 \tag{4b}$$

For some numbers a and b. In order to find the explicit relation between β and m, we ought to investigate the spectrum of the angular momentum further.

We know, that for some value m_{max} and m_{min} :

$$L_+|\beta, m_{max}\rangle = 0 \tag{5a}$$
$$L_-|\beta, m_{min}\rangle = 0 \tag{5b}$$

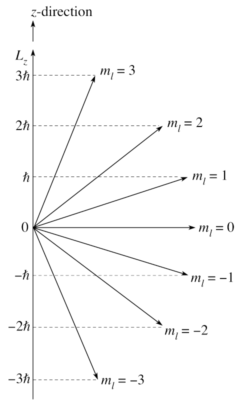

Figure 1: The z-component of the angular momentum is quantised

since the angular momentum will be totally alight with the wither z or $z-$ after successive application of L_+ or L_-. If we let $\ell\hbar$ be the total angular momentum eigenvalue, then obviously $m_{max} = \ell$ and $m_{min} = -\ell$.
Now we analyse (5b) further :

$$\begin{aligned}
\langle \beta, m_{max}| L_+^\dagger L_+ |\beta, m_{max}\rangle &= 0 \\
\langle \beta, m_{max}| (L_1 - iL_2)(L_1 + iL_2) |\beta, m_{max}\rangle &= 0 \\
\langle \beta, m_{max}| L_1^2 + L_2^2 + i[L_1, L_2] |\beta, m_{max}\rangle &= 0 \\
\langle \beta, m_{max}| L^2 - L_3^2 - L_3 |\beta, m_{max}\rangle &= 0 \\
\beta - \hbar^2 m_{max} - \hbar m_{max} &= 0 \\
\Rightarrow \beta = \hbar^2 \ell(\ell+1) \qquad (6)
\end{aligned}$$

Hence, we may denote the eigenstates in terms of ℓ instead of β, which is more physically relevant :

$$|\beta, m\rangle \longleftrightarrow |\ell, m\rangle$$

Such that:

$$L^2 |\ell, m\rangle = \hbar^2 \ell(\ell+1) \qquad (7)$$

Hence the magnitude of the angular momentum observable :

$$\langle L \rangle = \hbar\sqrt{\ell(\ell+1)} \qquad (8)$$

We may now write a full description for the angular momentum spectrum:

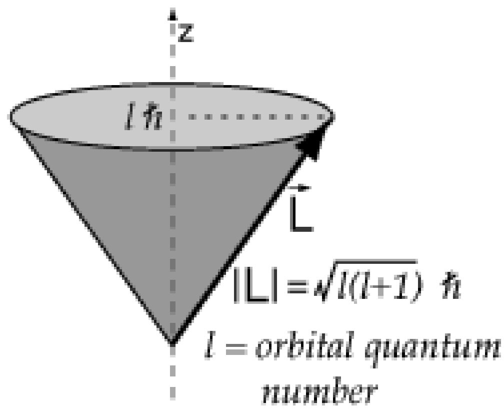

Figure 2: A vector model of the orbital quantum number

1. The **orbital** angular momentum eigenvalue is ℓ, it refers to the maximum positive or negative value the orbital angular momentum can take.

2. The z-component of the **orbital** angular momentum is m , sometimes , when other angular momenta are included we refer to it by m_ℓ takes the **integer** values between $+\ell$ and $-\ell$.

3. The length of the angular momentum is $\hbar\sqrt{\ell(\ell+1)}$.

4. We call ℓ the orbital / azimuthal quantum number and m_ℓ the magnetic quantum number.

5. There are other types of angular momenta, that shall be explored later, same analysis will be applied to them.