PHYS-505

The conservation laws in Quantum Mechanics

Lecture-6

The conservation laws and the symmetries

- *"In classical physics there are a number of quantities which are conserved —such as momentum, energy, and angular momentum. Conservation theorems about corresponding quantities also exist in quantum mechanics. The most beautiful thing of quantum mechanics is that the conservation theorems can, in a sense, be derived from something else, whereas in classical mechanics they are practically the starting points of the laws. (There are ways in classical mechanics to do an analogous thing to what we will do in quantum mechanics, but it can be done only at a very advanced level.) In quantum mechanics, however, the conservation laws are very deeply related to the principle of superposition of amplitudes, and to the symmetry of physical systems under various changes."* (R. Feynmann)

The conservation laws and the symmetries

- **Definition:** *A quantum mechanical quantity A is called conserved if its average value, at any of its states, remains constant with time.*
- In quantum mechanics the time evolution of the average of a physical quantity *A* is given by

$$i\hbar\frac{d\langle A\rangle}{dt}=\langle[A,H]\rangle+i\hbar\left\langle\frac{\partial A}{\partial t}\right\rangle$$

The conservation laws and the symmetries

- Thus for a conserved quantity (where A does not depend on time):

$$\left[A,\ H\right]=0$$

- We can easily prove that

$$\left[A^n,\ H\right]=0 \qquad (n\in\mathbf{N})$$

- A physical quantity which is conserved in classical physics is expected to be conserved in quantum physics as well.

The conservation laws and the symmetries

- The energy of a system is conserved **only** in the case where the potential is **time independent**.
- In the case where the potential depends on time we have for the time evolution of the Hamiltonian:

$$\frac{d\langle H\rangle}{dt} = \left\langle \frac{\partial H}{\partial t} \right\rangle = \left\langle \frac{\partial V}{\partial t} \right\rangle$$

The conservation laws and the symmetries

- The time independence of the external potential is equivalent to a *symmetry in time translation*. This is because at any time you observe the field it is the same.
- So: *conservation of energy is related to the symmetry in time translation.*
- In a closed physical system, where the energy is conserved, the symmetry in time translation is obligatory! This is known as **time homogeneity principle**. For a closed, isolated system there is no meaning to talk about the beginning of time.

The conservation laws and the symmetries

- We start with the commutation relations for the components of the momentum:

$$\left[p_i, H \right] = -i\hbar \frac{\partial H}{\partial x_i} = -i\hbar \frac{\partial V}{\partial x_i} = i\hbar F_i \qquad (i = x, y, z)$$

- This is the well-known result from classical physics - a momentum component is conserved if the corresponding force component is zero.

The conservation laws and the symmetries

- But to have a force component zero this means that the potential does not depend on this component. This means that all the points that come as a result of a parallel translation along the direction of the specific component are physically identical. In other words the system has a *translation symmetry along this direction.*
- **Conclusion:** *If a system has a translation symmetry along a direction, then the corresponding momentum component is conserved*

The conservation laws and the symmetries

- For an isolated system of particles the total momentum is always conserved. But this discussion has another interesting dimension. It is related to the *homogeneity of space*. Let's see the following example.
- Let's consider a **closed** system of two particles (which for simplicity are limited to move along x-axis). The homogeneity of space says that the potential interaction $V(x_1, x_2)$ between them cannot depend on the absolute positions of the particles x_1, x_2 but only on their relative position $x = x_1 - x_2$ i.e.

$$V(x_1, x_2) = V(x_1 - x_2) = V(x)$$

The conservation laws and the symmetries

- We can prove then that for the forces acting on the two particles F_1, F_2 we have:

$$F_1 = -F_2$$

- This is the famous *action-reaction principle* or the 3rd Newton's law!
- This law is a result of the homogeneity of space!
- **Conclusion:** *The translation symmetry is a consequence of space homogeneity*

The conservation of angular momentum

- We can prove for the components of the angular momentum the following relation:

$$\left[l_i, H \right] = i\hbar \tau_i \qquad (i = x, y, z)$$

Where τ_i are the components of the torque $\vec{\tau} = \mathbf{r} \times \mathbf{F}$ exerted on the particle.

Conclusion: *A component of the angular momentum is conserved only if the corresponding torque component is zero.*

The conservation laws and the symmetries

- Let's assume the direction along z-axis and let's consider that the torque component along z-direction is zero. In this case the corresponding force component intersects the z-axis. But this can happen only if the equipotential surfaces are symmetrical around z axis. In other words if the potential has a rotational symmetry around z-axis.
- In the special case where the potential is central it has full rotational symmetry and thus all of the angular momentum components are conserved.

The conservation laws and the symmetries

- In vacuum space the three directions are physically equivalent (*principle of the isotropy of space*).
- This principle tells us that the potential interaction

$$V(\mathbf{r}_1,\ \mathbf{r}_2) = V(\mathbf{r}_1 - \mathbf{r}_2) = V(\mathbf{r})$$

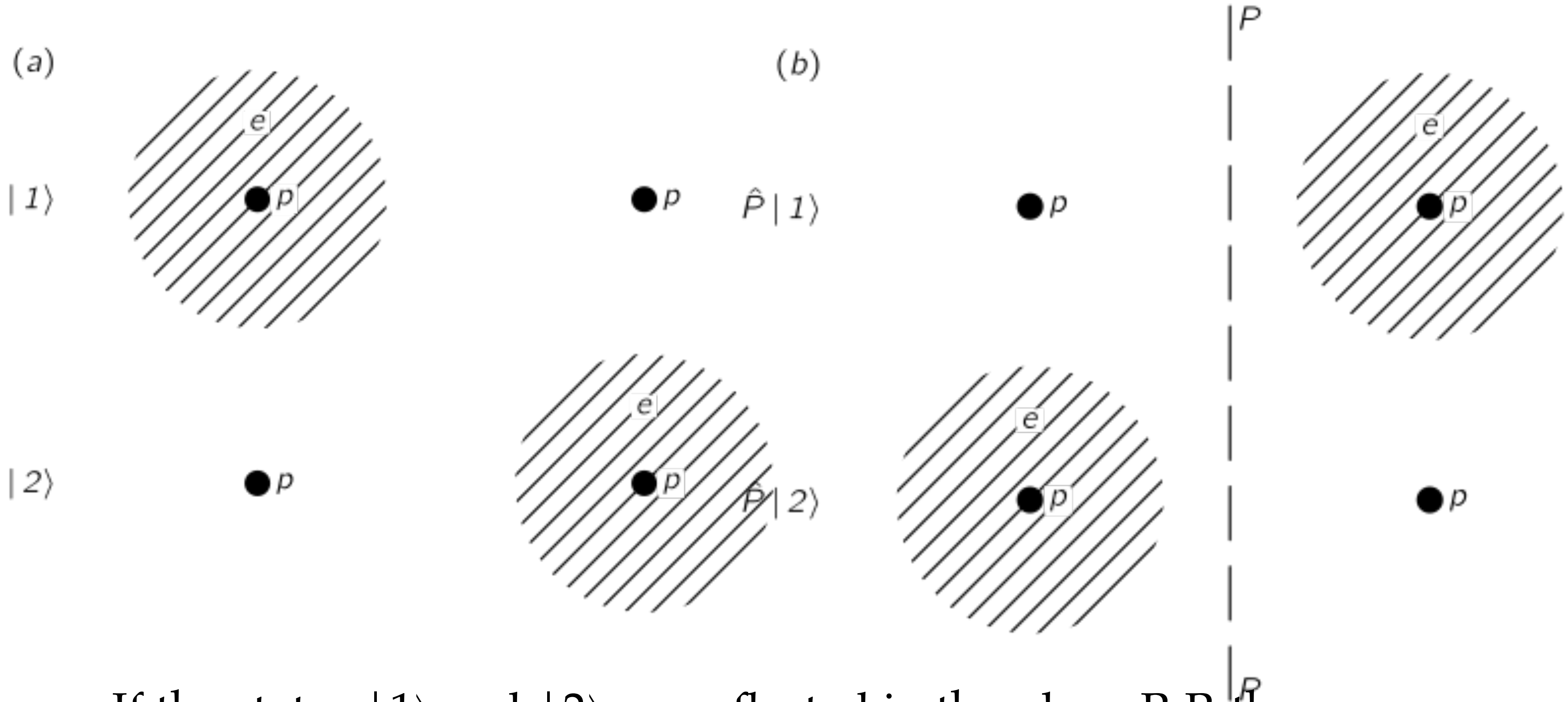
cannot depend on the orientation of the vector **r** but only on its magnitude, i.e., to be central

$$V = V(|\mathbf{r}_1 - \mathbf{r}_2|) = V(r)$$

The conservation laws and the symmetries

- **Conclusion:** *In a closed system the conservation of angular momentum is a consequence of the isotropy of space.*

Symmetry: The hydrogen molecular ion



If the states $|1\rangle$ and $|2\rangle$ are reflected in the plane P-P, they go into $|2\rangle$ and $|1\rangle$, respectively.

Symmetry: The hydrogen molecular ion

- The reflection of the system can be represented by an operator **P** which has the following properties:

$$\mathbf{P}\left|1\right\rangle = \left|2\right\rangle, \quad \mathbf{P}\left|2\right\rangle = \left|1\right\rangle$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Now we suppose that the the physics of the whole hydrogen molecular ion system is **symmetrical**.

Symmetry: The hydrogen molecular ion

At t=0

$$|\psi\rangle = |1\rangle$$

$$U(15,0)|1\rangle$$

At t=15s

$$|\psi\rangle = \sqrt{2/3}|1\rangle + i\sqrt{1/3}|2\rangle$$

At t=0

$$|\psi\rangle = |2\rangle$$

$$U(15,0)|1\rangle$$

At t=15s

$$|\psi\rangle = \sqrt{2/3}|2\rangle + i\sqrt{1/3}|1\rangle$$

A physical system is *symmetric* with respect to an operation when the operator of this commutes with *U*, the operation of the passage of time.

Symmetry

- Incidentally, since for infinitesimal times ε we have

$$U = 1 - iH\varepsilon / \hbar$$

we can see the above discussion implies that:

$$UH = HU \Rightarrow \left[U, H\right] = 0$$

Symmetry and Conservation

- We can show that an operation *which is a symmetry operation of the system* produces only a multiplication by a certain phase, and then you know that the same property will be true of the final state—the same operation multiplies the final state by the same phase factor.
- That's the basis of all the conservation laws of quantum mechanics.

Translation operator

- In quantum mechanics we have learned to associate a **unitary operator** U with an operation like translation or rotation. We have learned that for symmetry operations that differ infinitesimally from the identity transformation, we can write

$$U = 1 - iG\varepsilon / \hbar$$

- Where G is the Hermitian generator of the symmetry operator under question. Let us now suppose that the Hamiltonian H is invariant under U.

Translation operator

- Then this is equivalent to the following:

$$U^{+}HU = H \Rightarrow \left[G, H\right] = 0$$

- Hence G is a constant of motion. For instance, if H is invariant under translation, then momentum is a constant of the motion; if H is invariant under rotation, then angular momentum is a constant of the motion.