Continuous Random Variable ***X***:

The continuous random variable ***X*** has its values in an interval $X \in (a,b)$ and it has a probability distribution function (or a probability density function) (***p.d.f***) ***f(x)*** satisfies:

$$\forall x \in (a,b) \Rightarrow f(x) \geq 0 \qquad \& \qquad \int_a^b f(x)\, dx = 1$$

Which does mean that the total area under the curve of ***f(x)*** is 1. And the probabilities concerning X are achieved by:

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)\, dx$$

Which represents the area under the curve of ***f(x)*** and between (x_1, x_2) .

We define the mathematical Expectation (Or the expected value or the mean value) of X as:

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

While the variance of X is defined as:

$$\sigma^2 = V(x)\ \ E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = E(X^2) - \mu^2 \ \left(where \quad E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx \right)$$

The standard deviation of ***X*** is the positive square root of its variance $\sigma = \sqrt{\sigma^2}$.

Some Common Continuous Random Variables We Will Consider:

There are many different continuous probability distributions such as:

(1)The Normal distribution (2) The Student distribution

(3)The Chi-Square distribution (4) Fisher Probability distribution

The Normal Probability Distribution

The random variable X is said to have a Normal probability distribution with the two parameters μ and σ^2 (or $X \sim N(\mu, \sigma^2)$) if its ***p.d.f*** ***f (x)*** having the form :

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \;; \quad -\infty < x, \mu < \infty \;\; and \;\; 0 < \sigma < \infty$$

Clearly $\forall x \in (-\infty, \infty) \Rightarrow f(x) \geq 0$ & $\int_{-\infty}^{\infty} f(x)\, dx = 1 \cdot$

And this ***p.d.f*** satisfies the important properties:

(1) The curve of ***f(x)*** is bell – shaped and SYMMETRIC around μ

(2)It may be proven that: $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \mu$ & $V(X) = E(X^2) - \{E(X)\}^2 = \sigma^2$

(3) $f(x) \to 0 \;\; as \;\; X \to \pm\infty$

The normal distribution depends on the two parameters μ and σ^2. Where μ determines the location of the curve. But, σ determines the scale of the curve, i.e. the degree of flatness or Peaked ness of the curve**.** Moreover the normal variable with $(\mu = 0 \,,\, \sigma^2 = 1)$ is called standard normal and is given the symbol $Z \sim N(0,1)$, then;

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \;; \quad -\infty < Z < \infty \quad \& \quad \int_{-\infty}^{\infty} f(z)\, dz = 1$$

$$f(z) \to 0 \;\; as \;\; Z \to \pm\infty$$

$$E(Z) = \int_{-\infty}^{\infty} z f(z)\, dz = \mu_z = 0 \quad \& \quad \sigma^2 = V(Z) = E(Z^2) - \{E(Z)\}^2 = 1$$

Now, to evaluate each of the probabilities:

$$P(X < b) = P(X \leq b) = \int_{-\infty}^{b} f(x)dx \quad \& \quad P(X > a) = P(X \geq a) = \int_{a}^{\infty} f(x)dx$$

$$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b) = \int_{a}^{b} f(x)dx$$

We may use the standard linear transformation to find the required probabilities as follows:

$$P(X < b) = P(Z < \frac{b-\mu}{\sigma}) = P(Z < k) \quad (k = \frac{b-\mu}{\sigma})$$

Note that:

(1) P (Z > 0) = P (Z < 0) = 0.5

(2) If $Z \sim N(0,1)$ then:

i- P (Z < 2) = 0.9772 is the area under the Z-curve to the left to 2.

ii- P(-2.55 < Z < 2.55) = 0.9946 – 0.0054 = 0.9892 is the area between -2.55 and 2.55.

iii- P(-2.74 < Z < 1.53) =0.9370 – 0.0031 = 0.9339 is the area between -2.74 and 1.53.

iv- P(Z > 2.71)= 1 – 0.9966 = 0.0034 is the area under the Z-curve to the right of 2.71.

v- P(Z = 0.84) = 0

vi- It is meant by Z_A that $P(Z > Z_A) = A$. For instance:

$$Z_{0.10} = 1.28, \; Z_{0.05} = 1.645, \; Z_{0.025} = 1.96, \; Z_{0.01} = 2.33, Z_{0.005} = 2.575$$

Normal Distribution Applications

Example:

The 'Uptime 'is a custom-made light weight battery-operated activity monitor that records the amount of time an individual spend the upright position. In a study of children ages 8 to 15 years. The researchers found that the amount of time children spend in the upright position followed a normal distribution with mean of 5.4 hours and standard deviation of 1.3.Find if a child selected at random, then:

1-The probability that the child spend less than 3 hours in the upright position 24-hour period

$$P(X < 3) = P\left(Z < \frac{3-5.4}{1.3} = -1.85\right) = 0.0322$$

2-The probability that the child spend more than 5 hours in the upright position 24-hour period

$$P(X > 5) = P\left(Z > \frac{5-5.4}{1.3} = -0.308 \approx -0.31\right) = 1 - P(Z < -0.31) = 1 - 0.3520 = 0.648$$

3-The probability that the child spend exactly 6.2 hours in the upright position 24-hour period is

$P(X = 6.2) = 0$.

4-The probability that the child spend from 4.5 to 7.3 hours in the upright position 24-hour period

$$P(4.5 < X < 7.3) = P\left(\frac{4.5-5.4}{1.3} < Z < \frac{7.3-5.4}{1.3}\right)$$

$$= P(-0.69 < Z < 1.46) = P(Z<1.46) - P(Z< -0.69) = 0.9279 - 0.2451 = 0.6828.$$

Examples:

The brain weights of a certain population of adults follow approximately normal distribution with mean 1,400 gm and standard deviation 100 gm. What the percentage of the brain weights are:

(a) 1,500 gm or less?

(b) Between 1,325 and 1,500 gm?

(c) 1,325 or more?

(d) 1,475 gm or more?

(e) Between 1,475 gm and 1,600 gm?

Solution:

Assume a random variable X = weights of a certain population of adults; $X \sim N(1{,}400, 100^2)$; to find each of the required:

(a) $P(X \le 1{,}500) = P(Z \le \frac{1{,}500 - 1{,}400}{100}) = P(Z \le 1) = 0.8159$

And so the required percentage is 81.59 %.

(b) $P(1{,}325 \le X \le 1{,}500) = P(\frac{1{,}325-1{,}400}{100} \le Z \le \frac{1{,}500-1{,}400}{100}) = P(-1.75 \le Z \le 1)$

$= 0.8159 - 0.0401 = 0.7758$

(c) $P(X > 1{,}325) = P(Z > \frac{1{,}325-1{,}500}{100} = -1.75) = 1 - P(Z < -1.75) = 0.9599$

The required percentage will be 95.99%

(d) $P(X \ge 1{,}475) = P(Z \ge \frac{1{,}475-1{,}400}{100} =) = P(Z \ge 0.75) = 1 - P(Z < 0.75) = 1 - 0.7734 = 0.2266$

The required percentage is 22.66%

(e) $P(1{,}475 \le X \le 1{,}600) = P(\frac{1{,}475-1{,}400}{100} \le Z \le \frac{1{,}600-1{,}400}{100}) = P(0.75 \le Z \le 2)$

$= 0.9772 - 0.7734 = 0.2038$

The required percentage is 20.38%.

EXERCISE

Q1. (A) Suppose that Z is distributed according to the standard normal distribution.

1) The area under the curve to the left of $z = 1.43$ is:

(A) 0.0764 (B) 0.9236 (C) 0 (D) 0.8133

2) The area under the curve to the left of z = 1.39 is:

(A) 0.7268 (B) 0.9177 (C) .2732 (D) 0.0832

3) The area under the curve to the right of $z = -0.89$ is:

(A) 0. 7815 (B) 0.8133 (C) 0.1867 (D) 0.0154

4) The area under the curve between $z = -2.16$ and $z = -0.65$ is:

(A) 0.7576 (B) 0.8665 (C) 0.0154 (D) 0.2424

5) The value of k such that $P(0.93 < Z < k) = 0.0427$ is:

(A) 0.8665 (B) −1.11 (C) 1.11 (D) 1.00

(B) Suppose that Z is distributed according to the standard normal distribution. Find:

1) P(Z< −3.9) 2) P(Z> 4.5) 3) P(Z< 3.7) 4) P(Z> −4.1)

Q2. The finished inside diameter of a piston ring is normally distributed with a mean of 12 centimeters (***c.m***) and a standard deviation of 0.03 centimeter. Then,

1) The proportion of rings that will have inside diameter less than 12.05 ***c.m*** is:

(A) 0.0475 (B) 0.9525 (C) 0.7257 (D) 0.8413

2) The proportion of rings that will have inside diameter exceeding 11.97 ***c.m*** is:

(A) 0.0475 (B) 0.8413 (C) 0.1587 (D) 0.4514

3) The probability that a piston ring will have an inside diameter between 11.95 and 12.05 ***c.m*** is:

(A) 0.905 (B) −0.905 (C) 0.4514 (D) 0.7257

Q3. The average life of a certain type of small motor is 10 years with a standard deviation of 2 years. Assume the live of the motor is normally distributed. The manufacturer replaces free all motors that fail while under guarantee. If he is willing to replace only 1.5% of the motors that fail, then he should give a guarantee of:

(A) 10.03 years (B) 8 years (C) 5.66 years (D) 3 years

Q4. A machine makes bolts (that are used in the construction of an electric transformer). It produces bolts with diameters (X) following a normal distribution with a mean of 0.060 inches and a standard deviation of 0.001 inches. Any bolt with diameter less than 0.058 inches or greater than 0.062 inches must be scrapped. Then

(1) The proportion of bolts that must be scrapped is equal to

(A) 0.0456 (B) 0.0228 (C) 0.9772 (D) 0.3333 (E) 0.1667

(2) If P(X>a)= 0.1949, then a equals to:

(A) 0.0629 (B) 0.0659 (C) 0.0649 (D) 0.0669 (E) 0.0609

Q5. The diameters of ball bearings manufactured by an industrial process are normally distributed with a mean μ = 3.0 cm and a standard deviation σ = 0.005 cm. All ball bearings with diameters not within the specifications $\mu \pm d$ cm (d > 0) will be scrapped.

(1) Determine the value of d such that 90% of ball bearings manufactured by this process will not be scrapped.

(2) If d = 0.005, what is the percentage of manufactured ball bearings that will be scraped?

Q6. The weight of a large number of fat persons is nicely modeled with a normal distribution with mean of 128 kg and a standard deviation of 9 kg.

(1) The percentage of fat persons with weights at most 110 kg is

(A) 0.09 % (B) 90.3 % (C) 99.82 % (D) 2.28 %

(2) The percentage of fat persons with weights more than 149 kg is

(A) 0.09 % (B) 0.99 % (C) 9.7 % (D) 99.82 %

(3) The weight x above which 86% of those persons will be

(A) 118.28 (B) 128.28 (C) 154.82 (D) 81.28

(4) The weight x below which 50% of those persons will be

(A) 101.18 (B) 128 (C) 154.82 (D) 81

Q7. The random variable X, representing the lifespan of a certain electronic device, is normally distributed with a mean of 40 months and a standard deviation of 2 months. Find

1)$P(X<38)$. 2)$P(38<X<40)$. 3)$P(X=38)$.

4)The value of x such that $P(X<x)=0.7324$. (41.24)

Q8. If the random variable X has a normal distribution with the mean μ and the variance σ^2, then $P(X<\mu+2\sigma)$ equals to

(A) 0.8772 (B) 0.4772 (C) 0.5772 (D) 0.7772 (E) 0.9772

Q9. If the random variable X has a normal distribution with the mean μ and the variance 1, and if $P(X<3)=0.877$, then μ equals to

(A) 3.84 (B) 2.84 (C) 1.84 (D) 4.84 (E) 8.84

Q10. Suppose that the marks of the students in a certain course are distributed according to a normal distribution with the mean 70 and the variance 25. If it is known that 33% of the student failed the exam, then the passing mark x is

(A) 67.8 (B) 60.8 (C) 57.8 (D) 50.8 (E) 70.8

Q11. If the random variable X has a normal distribution with the mean 10 and the variance 36, then

1. The value of X above which an area of 0.2296 lie is

(A) 14.44 (B) 16.44 (C) 10.44 (D) 18.44 (E) 11.44

2. The probability that the value of X is greater than 16 is

(A) 0.9587 (B) 0.1587 (C) 0.7587 (D) 0.0587 (E) 0.5587

Q12. Suppose that the marks of the students in a certain course are distributed according to a normal distribution with the mean 65 and the variance 16. A student fails the exam if he obtains a mark less than 60. Then the percentage of students who fail the exam is

(A) 20.56% (B) 90.56% (C) 50.56% (D) 10.56% (E)40.56%

Q13. The average rainfall in a certain city for the month of March is 9.22 centimeters. Assuming a normal distribution with a standard deviation of 2.83 centimeters, then the probability that next March, this city will receive:

(1) Less than 11.84 centimeters of rain is:

(A) 0.8238 (B) 0.1762 (C) 0.5 (D) 0.2018

(2) More than 5 centimeters but less than 7 centimeters of rain is:

(A) 0.8504 (B) 0.1496 (C) 0.6502 (D) 0.34221

(3) More than 13.8 centimeters of rain is:

(A) 0.0526 (B) 0.9474 (C) 0.3101 (D) 0.4053

<u>The Chi square Probability Distribution</u>

The random variable ***X***= χ^2 is said to have a chi-square probability distribution with the parameter or the degrees of freedom (***d.f***) ***υ*** if its ***p.d.f***. ***f (x)*** is:

$$f(x) = \frac{1}{2\Gamma(\frac{\nu}{2})}\left(\frac{x}{2}\right)^{\left(\frac{\nu}{2}-1\right)} e^{-(\frac{x}{2})} \quad , \quad 0 < x < \infty .$$

Which does mean that there are so many chi-square distributions depends on the ***d.f*** =υ. The chi-square distribution satisfies the following properties:

$$f(x) \geq 0; \quad 0 < x < \infty \quad \& \quad \int_0^\infty f(x)\, dx = 1$$

This ***p.d.f***. satisfies the important properties:

(1) The curve of (χ^2) is not bell – shaped but it is skewed curve to the right depends on ***υ***.

(2)We may prove that the expected value and the variance of χ^2 are:

$$E(\chi^2)=\nu \quad \& \quad V(\chi^2) = 2\nu$$

We may denote by $\chi^2_{\nu,\alpha}$ to the critical point χ^2 that satisfies $P(\chi^2 > \chi^2_{\nu,\alpha})=\alpha$.

Eventually, if a random normal sample of size ***n*** and σ^2 represents the population variance then

$\dfrac{(n-1)\,S^2}{\sigma^2}$ is χ^2 with $\nu=(n-1)$ where S^2 represents the sample variance.

<u>Examples</u>:

1. Determine the value of $\chi^2_{(15\,,\,0.005)}$ which satisfies $P(\chi^2 \geq \chi^2_{15\,,\,0.005})=0.005$.

The required is that tabulated chi-square value which leaves an area of α =0.005 to the right with the ***d.f=*** 15. From the table we get:

$$\chi^2_{(15\,,\,0.005)} = 32.801.$$

2. If $P(\chi^2 < \chi^2_{20}) = 0.99$ .So he asked us to find the value of χ^2_{20} which leaves an area of 0.99 to the left with ***d.f*** = 20 …. Or

$$P(\chi^2 > \chi^2_{20,0.01}) = 1 - P(\chi^2 < \chi^2_{20}) = 0.01 \Rightarrow \chi^2_{20,0.01} = 37.7663.$$

3. Find the probability $P(\chi^2_{20\,,\,0.99} \leq \chi^2 \leq \chi^2_{20\,,\,0.01})$. The required probability value equals 0.99 – 0.01= 0.98.

<u>The t- Probability Distribution (The Student distribution)</u>

The Student distribution t with degree of freedom υ is bell – shaped and symmetric as $N(0,1)$ but is less peaked in the center $\mu_t = 0$ and higher tails $f(t) \xrightarrow[t \to \pm\infty]{} 0$ depends on the υ where

$V(T) = \sigma_t^2 = \dfrac{\nu}{\nu - 2} > 1$. The critical point $t_{\nu,\alpha}$ is $P(T > t_{\nu,\alpha}) = \int_{t_{\nu,\alpha}}^{\infty} f(t)dt = \alpha$. The

distribution t approaches to $N(0,1)$ as the number of degrees of freedom υ approaches ∞**.** But in practice t is approximated by $N(0,1)$ as $n \geq 30$.

It can be shown that:

1- If $Z \sim N(0,1)$ & $Y \sim \chi^2_{(\nu)} \Rightarrow T_\nu = Z / \sqrt{\dfrac{Y}{\nu}}$

2- If $X_1, X_2, ..., X_n \sim N(\mu, \sigma^2) \Rightarrow T_{(n-1)} = \dfrac{\overline{X} - \mu}{S / \sqrt{n}}$

<u>Examples</u>

(1) If $P(T > t_{\nu,\alpha}) =$ α where υ=7, α=0.975 then $t_{7,0.975} = -2.3646$

(2) If $P(T > t_{\nu,\alpha}) =$ α where υ=24, α =0.995 then $t_{24,0.975} = -2.3646$

(3) If $P(T > t_{18,0.975}) = 0.975$ then $t_{18,0.975} =$ - 2.1009

(4) If $P(T < t_{22}) = 0.99$ So he asked us to find the value of t such that

$P(T < t_{22}) = 0.99$ *and so* $P(T > t_{22,0.01}) = 0.01$ *i.e* $t_{22,0.01} = 2.508$

<u>The F-distribution (Or Fisher probability distribution)</u>

F is said to have the F-distribution (Or Fisher probability distribution) with the two degrees υ_1 and υ_2 if its probability density function of the form:

$$f(F)=\frac{1}{\beta(\frac{\nu_1}{2},\frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{(\frac{\nu_1}{2})} F^{\frac{\nu_1}{2}-1}\left(1+\frac{\nu_1}{\nu_2}F\right)^{-(\nu_1+\nu_2)/2} \quad ; 0<F<\infty$$

Which satisfies:

(1) The curve of f(F) is not bell – shaped but it is skewed curve to the right depends on the two parameters υ_1 and υ_2.

(2)We may prove that the expected value of the random variable T is

$$E(F)=\int_{-\infty}^{\infty} F\ f(F)\ dF=\frac{\nu_1}{\nu_2-2} \quad \& \quad V(F)=\frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)} \quad ;\ \nu_2>4$$

We denote the critical point $F_{\nu_1,\nu_2,\alpha}$: $P(F > F_{\nu_1,\nu_2,\alpha})=\int_{F_{\nu_1,\nu_2,\alpha}}^{\infty} f(F)dF = \alpha$ and we accept that $F_{\nu_1,\nu_2,\alpha}=\frac{1}{F_{\nu_2,\nu_1,1-\alpha}}$.

(3) It can be shown that: $X \sim \chi^2_{(\nu_1)} \quad \& \quad Y \sim \chi^2_{(\nu_2)} \Rightarrow F_{\nu_1,\nu_2}=\frac{X/\nu_1}{Y/\nu_2}$;

<u>Examples</u>

(1) If $P(F \geq F_{10,8,0.05})=0.05 \Rightarrow F_{10,8,0.05}=3.35$.

(2) Find the F-value which satisfies: $P(F < F_{10,8})=0.05$:

The required F value is that critical tabulated F value which leaves an area of 0.05 under the curve to the left. That is F value which leaves an area of 0.95 to the right.

From the F-able at α = 0.05, we can say that: the required F value

$$F_{10,8,0.95}=\frac{1}{F_{8,10,0.05}}=\frac{1}{3.07}=0.3257=3.35$$

(3) Find the tabulated F-value of $F_{10,8,0.99}$. We can get the required F-value by using

$$F_{10,8,\ 0.99} = \frac{1}{F_{8,10\ ,0.01}} = \frac{1}{5.06} = 0.1976$$
