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Generalization of Posner's Theorems

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Abstract. In this paper we generalize Posner's first theorem to a 3-prime near-ring with a (σ,τ)-derivation. We prove that a prime ring with a non-zero (σ,τ)-derivation is commutative if $\sigma(x)d(x) = d(x)\tau(x)$ for all $x \in U$ where U is a suitable subset of R. Also, we generalize Posner's second theorem completely to a prime ring with a (σ,σ)-derivation and partially to a prime ring with a (σ,τ)-derivation.

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1. Introduction

Throughout this paper R will be a ring or a left near-ring. $Z(R)$ will be its multiplicative center and σ,τ two endomorphisms from R to R. We say that R is prime (3-prime for near-rings) if, for all $x,y \in R$, $xRy = \{0\}$ implies $x = 0$ or $y = 0$. We say that U is a semigroup right (left) ideal of R, if U is a non-empty subset of R satisfies $UR \subseteq U$ ($RU \subseteq U$). We say that U is a semigroup ideal if it is both a semigroup right and left ideal. For all $x,y \in R$, we write $[x,y] = xy - yx$ for the multiplicative commutator, $[x,y]_{\sigma,\tau} = \sigma(x)y - y\tau(x)$ and $(x,y) = x + y - x - y$ for the additive commutator. A map $d : R \to R$ is called a (σ,τ)-derivation if d is additive and $d(xy) = \sigma(x)d(y) + d(x)\tau(y)$ for all $x,y \in R$. If $\tau = 1_R$, then d is called a σ-derivation. If $\sigma = \tau = 1_R$, then d is the usual derivation. An element $x \in R$ is called a left (right) zero divisor in R if there exists a non-zero element $y \in R$ such that $xy = 0$ ($yx = 0$). A zero divisor is either a left or a right zero divisor. By an integral near-ring, we mean a near-ring without non-zero divisors of zero. A near-ring R is called a constant near-ring, if $xy = y$ for all $x,y \in R$ and is called a zero-symmetric near-ring, if $0x = 0$ for all $x \in R$. For any group $(G,+)$, $M_o(G)$ denotes the near-ring of all zero preserving maps from G to G with the two operations of addition and composition of maps. An abelian near-ring R is a near-ring such that $(R,+)$ is abelian. We refer the reader to the books of Meldrum [15] and Pilz [17] for basic results of near-ring theory and its applications.

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In this paper we use the commutator $[x,y]_{\sigma,\tau}$ to mean $\sigma(x)y - y\tau(x)$, but its usual form is $x\sigma(y) - \tau(y)x$ with using that $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$. According to the last form, Argac, Kaya and Kisir showed in [1] that a prime ring R admits a non-zero (σ,τ)-derivation such that $[d(x),x]_{\sigma,\tau} = 0$ for all $x \in I$ if and only if R is commutative and $\sigma = \tau$, where I is a non-zero right ideal of R. They also showed that a prime ring R of characteristic not 2 admits a non-zero (σ,τ)-derivation such that $[d(x),x]_{\sigma,\tau} \in C_{\sigma,\tau}$ for all $x \in I$ if and only if R is commutative and $\sigma = \tau$, where $C_{\sigma,\tau} = \{x \in R : x\sigma(y) = \tau(y)x$ for all $y \in R\}$. Also, Ashraf and Rehman showed in Theorem 1 in [2] that a 2-torsion free prime ring R is commutative if R admits a non-zero (σ,τ)-derivation such that $[d(x),x]_{\sigma,\tau} = 0$ for all $x \in R$. In [3], Aydin had extended that theorem to $[d(x),x]_{\sigma,\tau} \in C_{\sigma,\tau}$ for all $x \in R$. All above papers used that σ and τ are automorphisms on R. In the literature of studying commutativity of rings and near-rings, there are also some works studied the commutativity of rings and near-rings without the use of derivations, for example see [5] and [6]. Also, see [16] for subcommutativity in near-rings.

In Section 2 we give some well-known results and we add some new auxiliary results on a near-ring R admitting a non-zero (σ,τ)-derivation d, which will be useful in the sequel. In Section 3 we study the problem of Posner for the composition of two derivations, in the more general case the composition of a (σ,τ)-derivation and an (α,β)-derivation, where α is an automorphism and, σ, β and τ are epimorphisms on a near-ring R. Consequently, we generalize Posner's first theorem for (σ,τ)-derivations in Theorem 3.1 which generalizes results due to K. I. Beidar, Y. Fong and X. K. Wang; O. Golbasi and M. S. Samman.

Section 4 is devoted to study Posner's second theorem using (σ,τ)-derivations on prime rings. Consequently, we generalize Lemma 3 of [18] to (σ,τ)-derivations on prime rings. In Theorem 4.4 we study Posner's second theorem using (σ,τ)-derivations on prime rings. Theorem 4.5 is a generalization of Posner's second theorem to (σ,σ)-derivations on prime rings, where σ is an epimorphism on R. In the last of this section we study the condition $d(x^2) \in Z(R)$ for all $x \in R$, where d is a non-zero (σ,τ)-derivation on a prime ring R.

2. Preliminaries and some results

We need the following lemmas:

Lemma 2.1. [10, Lemma 1] *An additive mapping d on a near-ring R is a (σ,τ)-derivation if and only if $d(xy) = d(x)\tau(y) + \sigma(x)d(y)$, for all $x, y \in R$.*

Lemma 2.2. [10, Lemma 2] *Let R be a near-ring with a (σ,τ)-derivation d such that τ is an epimorphism. Then R satisfies the partial distributive law, $(\sigma(x)d(y) + d(x)\tau(y))c = \sigma(x)d(y)c + d(x)\tau(y)c$ and $(d(x)\tau(y) + \sigma(x)d(y))c = d(x)\tau(y)c + \sigma(x)d(y)c$ for all $x, y, c \in R$.*

Lemma 2.3. [7, Lemma 1.2(iii)] *Let R be a 3-prime near-ring and $x \in Z(R) - \{0\}$. If either yx or xy in $Z(R)$, then $y \in Z(R)$.*

Lemma 2.4. [9, Lemma 3(i),(ii)] *Let R be a 3-prime near-ring and $x \in Z(R) - \{0\}$. Then x is not a zero divisor in R.*

Lemma 2.5. [10, Lemma 3] *Let d be a non-zero (σ,τ)-derivation on a 3-prime near-ring R.*

(i) *If $d(R)x = \{0\}$ and τ is onto, then $x = 0$.*

(ii) *If* $xd(R) = \{0\}$, *R is zero-symmetric and* σ *is onto, then* $x = 0$.

Lemma 2.6. [13, Proposition 2.7] *A near-ring R is zero-symmetric if and only if R admits a* (σ,τ)*-derivation d such that* σ,τ *are endomorphisms and* τ *is either one-to-one or onto.*

Lemma 2.7. *Let R be a near-ring with a* (σ,τ)*-derivation d such that* $2R = \{0\}$ *and* σ,τ *commute with d. Then* d^2 *is a* (σ^2,τ^2)*-derivation on R.*

Proof. For all $x,y \in R$, we have $d^2(x+y) = d^2(x)+d^2(y)$ since d is an additive mapping on R. Now, for all $x,y \in R$ we get

$$\begin{aligned} d^2(xy) &= d(d(xy)) = d(\sigma(x)d(y)+d(x)\tau(y)) \\ &= \sigma^2(x)d^2(y)+d\sigma(x)\tau d(y)+\sigma d(x)d\tau(y)+d^2(x)\tau^2(y) \\ &= \sigma^2(x)d^2(y)+d\sigma(x)d\tau(y)+d\sigma(x)d\tau(y)+d^2(x)\tau^2(y) \\ &= \sigma^2(x)d^2(y)+2d\sigma(x)d\tau(y)+d^2(x)\tau^2(y) = \sigma^2(x)d^2(y)+d^2(x)\tau^2(y). \end{aligned}$$

Thus, $d^2(xy) = \sigma^2(x)d^2(y)+d^2(x)\tau^2(y)$ for all $x,y \in R$ and d^2 is a (σ^2,τ^2)-derivation on R. ∎

Lemma 2.8. [7, Lemma 1.3(iii)] *Let R be a 3-prime near-ring with a non-zero semigroup right ideal U of R. If there exists* $x \in R$ *which centralizes U, then* $x \in Z(R)$*. Moreover, if R is a prime ring and U is a semigroup left ideal, then* $x \in Z(R)$*.*

Lemma 2.9. [11, Lemma 4] *Let R be a 3-prime near-ring with a* (σ,τ)*-derivation d.*

(i) *If R is zero-symmetric and U is a non-zero semigroup right ideal of R such that* σ *is an epimorphism,* $\sigma(U) \neq \{0\}$ *and* $d(U) = \{0\}$*, then* $d = 0$.

(ii) *If U is a non-zero semigroup left ideal of R such that* τ *is an epimorphism,* $\tau(U) \neq \{0\}$ *and* $d(U) = \{0\}$*, then* $d = 0$.

Lemma 2.10. [7, Lemma 1.5] *Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U such that* $U \subseteq Z(R)$*. Then R is a commutative ring.*

Lemma 2.11. [7, Lemma 1.4] *Let R be a 3-prime near-ring with a non-zero semigroup ideal U. If* $x,y \in R$ *and* $xUy = \{0\}$*, then* $x = 0$ *or* $y = 0$.

Lemma 2.12. [13, Corollary 4.6] *Let R be a 3-prime near-ring with a non-zero* (σ,τ)*-derivation d such that one of* σ,τ *is either a monomorphism or an epimorphism. If* $d(R) \subseteq Z(R)$*, then R is a commutative ring.*

Lemma 2.13. [13, Theorem 5.4] *Let R be a 3-prime near-ring with a non-zero* (σ,τ)*-derivation d such that* τ *is an automorphism and* $d(xy) = d(yx)$ *for all* $x,y \in R$*. Then R is a commutative ring.*

Lemma 2.14. [13, Theorem 5.9] *Let R be a 3-prime near-ring with a non-zero* (σ,τ)*-derivation d such that* $d(xy) = -d(yx)$ *for all* $x,y \in R$*. If* τ *is an automorphism on R, then R is a commutative ring of characteristic 2.*

3. Posner's first theorem

In this section we generalize Posner's first theorem for (σ,τ)-derivations on near-rings. We need the following two lemmas to prove the first theorem in this section.

Lemma 3.1. *Let R be a near-ring with a (σ,τ)-derivation d and θ be any endomorphism of R. Then*

(i) *θd is a $(\theta\sigma,\theta\tau)$-derivation on R.*
(ii) *$d\theta$ is a $(\sigma\theta,\tau\theta)$-derivation on R.*

Proof. (i) Clearly the composition of two additive mappings on R is an additive mapping. Now, for all $x,y\in R$, we have $\theta d(xy)=\theta(d(xy))=\theta(\sigma(x)d(y)+d(x)\tau(y))=\theta\sigma(x)\theta d(y)+\theta d(x)\theta\tau(y)$ and then θd is a $(\theta\sigma,\theta\tau)$-derivation on R.

(ii) The proof is similar to (i). ∎

Lemma 3.2. *Let R be a near-ring with a non-zero (σ,τ)-derivation d. Suppose one of the following two conditions holds:*

(i) *R is a 3-prime near-ring and τ is onto, or*
(ii) *There exists $a\in R$ such that $d(a)$ is not a left zero divisor in R and τ is either one-to-one or onto.*

Then $nR=\{0\}$ if and only if $nd(R)=\{0\}$.

Proof. Clearly if $nR=\{0\}$, then $nd(R)=\{0\}$. Conversely, suppose $nd(R)=\{0\}$. Then $0=nd(b)=d(nb)$ for all $b\in R$. Now, for all $x,y\in R$

$$0=d(n(yx))=d(y(nx))=\sigma(y)d(nx)+d(y)\tau(nx)=d(y)\tau(nx).$$

If R is 3-prime and τ is onto, then $d(R)\tau(nx)=\{0\}$ implies $\tau(nx)=0$ for all $x\in R$ by Lemma 2.5(i). It follows that $\{0\}=\tau(nR)=n\tau(R)=nR$. If there exists $a\in R$ such that $d(a)$ is not a left zero divisor in R, then $d(a)\tau(nx)=0$ and then $\tau(nx)=0$ for all $x\in R$. Therefore $\tau(nR)=\{0\}$. If τ is onto, then by the same way above $nR=\{0\}$ and if τ is one-to-one, then $\tau(nR)=\{0\}$ implies $nR=\{0\}$. ∎

The conditions "τ is onto" in Lemma 3.2(i) and "τ is either one-to-one or onto" in Lemma 3.2(ii) are not redundant as the following example shows.

Example 3.1. Let $(R,+)$ be the additive abelian group $(\mathbb{Z}_4,+)$ and define the multiplication to make R a constant near-ring. Then R is 3-prime. Suppose $\tau=0$ and σ is any endomorphism on R, then any additive mapping d on R is a (σ,τ)-derivation. Define $d:R\to R$ by $d(\overline{x})=\overline{x}+\overline{x}$ for all $\overline{x}\in R$. Then $d(\overline{x}+\overline{y})=\overline{x}+\overline{y}+\overline{x}+\overline{y}=\overline{x}+\overline{x}+\overline{y}+\overline{y}=d(\overline{x})+d(\overline{y})$ for all $\overline{x},\overline{y}\in R$ and d is an additive endomorphism of R. So d is a (σ,τ)-derivation on R. Also, $d(\overline{1})=\overline{1}+\overline{1}=\overline{2}$ is not a left zero divisor in R by the definition of the multiplication. Observe that $d(2\overline{x})=d(\overline{x}+\overline{x})=\overline{x}+\overline{x}+\overline{x}+\overline{x}=4\overline{x}=\overline{0}$ for all $\overline{x}\in R$. Thus, $2d(R)=\{\overline{0}\}$. But $2R\neq\{\overline{0}\}$ as $2(\overline{1})=\overline{1}+\overline{1}=\overline{2}\neq\overline{0}$.

The following theorem generalizes Theorem 1.1 of [4], Theorem 2.5 of [11] and the main Theorem of [19].

Theorem 3.1. *Let R be a 3-prime near-ring with a (σ,τ)-derivation d and an (α,β)-derivation D such that α commutes with β, α is an automorphism, σ,β,τ are epimorphisms and α,β,τ commute with D. If dD is a $(\sigma\alpha,\tau\beta)$-derivation, then one of the following statements holds:*

(i) $d=0$
(ii) $D=0$
(iii) $2R=\{0\}$.

Proof. Since τ is an epimorphism, we have R is zero-symmetric by Lemma 2.6. As dD is a $(\sigma\alpha,\tau\beta)$-derivation, so $dD(ab)=\sigma\alpha(a)dD(b)+dD(a)\tau\beta(b)$ for all $a,b\in R$. On the other hand, d is a (σ,τ)-derivation and D is an (α,β)-derivation. Thus, $dD(ab)=d(\alpha(a)D(b)+D(a)\beta(b))=\sigma\alpha(a)dD(b)+d(\alpha(a))\tau D(b)+\sigma(D(a))d(\beta(b))+dD(a)\tau\beta(b)$. Comparing the previous two equations, we get

$$d(\alpha(a))\tau(D(b))+\sigma(D(a))d(\beta(b))=0 \quad \text{for all} \quad a,b\in R. \tag{3.1}$$

Replace a by ac where $c\in R$. So using the partial distributive law (Lemma 2.2), we have for all $a,b,c\in R$

$$\begin{aligned}0&=d(\alpha(ac))\tau D(b)+\sigma(D(ac))d(\beta(b))=d(\alpha(a)\alpha(c))\tau D(b)+\sigma(D(ac))d(\beta(b))\\&=d\alpha(a)\tau\alpha(c)\tau D(b)+\sigma\alpha(a)d\alpha(c)\tau D(b)+\sigma(\alpha(a)D(c)+D(a)\beta(c))d(\beta(b))\\&=d\alpha(a)\tau\alpha(c)\tau D(b)+\sigma\alpha(a)d\alpha(c)\tau D(b)+(\sigma\alpha(a)\sigma D(c)+\sigma D(a)\sigma\beta(c))d(\beta(b)).\end{aligned}$$

Notice that σD is a $(\sigma\alpha,\sigma\beta)$-derivation by Lemma 3.1. Since $\sigma\beta$ is onto, we can use the partial distributive law to obtain

$$\begin{aligned}0&=d\alpha(a)\tau\alpha(c)\tau D(b)+\sigma\alpha(a)d\alpha(c)\tau D(b)+\sigma\alpha(a)\sigma D(c)d(\beta(b))\\&\quad+\sigma D(a)\sigma\beta(c)d(\beta(b))\\&=d\alpha(a)\tau\alpha(c)\tau D(b)+\sigma\alpha(a)(d\alpha(c)\tau D(b)+\sigma D(c)d(\beta(b)))\\&\quad+\sigma D(a)\sigma\beta(c)d(\beta(b))\end{aligned}$$

for all $a,b,c\in R$. By using (3.1) with c instead of a, we get for all $a,b,c\in R$

$$d\alpha(a)\tau\alpha(c)\tau D(b)+\sigma D(a)\sigma\beta(c)d(\beta(b))=0. \tag{3.2}$$

As α is bijective, we obtain $d\alpha(a)\tau(r)\tau D(b)+\sigma D(a)\sigma\beta(\alpha^{-1}(r))d(\beta(b))=0$ for all $a,b,r\in R$ where $r=\alpha(c)$. Taking $r=D(t)$ where $t\in R$, we obtain $d\alpha(a)\tau D(t)\tau D(b)+\sigma D(a)\sigma\beta\alpha^{-1}D(t)d(\beta(b))=0$ for all $a,b,t\in R$. Since $\beta\alpha^{-1}$ commutes with D, we have

$$d\alpha(a)\tau D(t)\tau D(b)+\sigma D(a)\sigma(D(\beta\alpha^{-1}(t))d(\beta(b))=0. \tag{3.3}$$

Replacing a by $\beta\alpha^{-1}(t)$ in equation (3.1), we deduce that $\sigma(D(\beta\alpha^{-1}(t))d(\beta(b))=-d(\alpha(\beta\alpha^{-1}(t)))\tau D(b)$. Since α and β commute, we have $\sigma(D(\beta\alpha^{-1}(t))d(\beta(b))=-d(\beta(t))\tau D(b)$ for all $t,b\in R$. Therefore, (3.3) becomes $0=d\alpha(a)\tau D(t)\tau D(b)+\sigma D(a)(-d(\beta(t))\tau D(b))$ which means

$$d\alpha(a)\tau D(t)\tau D(b)=\sigma D(a)d(\beta(t))\tau D(b) \quad \text{for all} \quad a,b,t\in R. \tag{3.4}$$

Replacing b by tk in (3.1) where $t,k\in R$, we have

$$\begin{aligned}0&=d(\alpha(a))\tau D(tk)+\sigma(D(a))d(\beta(tk))=d(\alpha(a))\tau D(tk)+\sigma(D(a))d(\beta(t)\beta(k))\\&=d\alpha(a)\tau(D(t)\beta(k)+\alpha(t)D(k))+\sigma D(a)(\sigma\beta(t)d\beta(k)+d\beta(t)\tau(\beta(k)))\\&=d\alpha(a)\tau D(t)\tau(\beta(k))+d\alpha(a)\tau\alpha(t)\tau D(k)+\sigma D(a)\sigma\beta(t)d\beta(k)+\sigma D(a)d\beta(t)\tau(\beta(k))\\&=d\alpha(a)\tau D(t)\tau(\beta(k))+\sigma D(a)d\beta(t)\tau(\beta(k))\end{aligned}$$

as $d\alpha(a)\tau\alpha(t)\tau D(k)+\sigma D(a)\sigma\beta(t)d\beta(k)=0$ by (3.2). Then $d\alpha(a)\tau D(t)\tau(r)+\sigma D(a)d\beta(t)\tau(r)=0$ for all $a,t,r\in R$, since β is onto. Taking $r=D(b)$ where $b\in R$ in the last equation, we obtain

$$d\alpha(a)\tau D(t)\tau D(b)+\sigma D(a)d\beta(t)\tau D(b)=0 \quad \text{for all} \quad a,b,t\in R. \tag{3.5}$$

Substituting (3.4) in (3.5) and using $\tau D = D\tau$, we get for all $a,b,t \in R$

$$0 = d(\alpha(a))D\tau(t)D\tau(b) + d(\alpha(a))D\tau(t)D\tau(b) = d(\alpha(a))D(\tau(t))(2D(\tau(b))).$$

Since α and τ are onto, we have $d(R)D(R)(2D(R)) = \{0\}$. Suppose $d \neq 0$. So $D(R)(2D(R)) = \{0\}$ by Lemma 2.5(i). If $D \neq 0$, then $2D(R) = \{0\}$ by Lemma 2.5(i) and hence $2R = \{0\}$ by Lemma 3.2(i) ∎

The following corollary generalizes [20, Corollary 1].

Corollary 3.1. *Let R be a 3-prime near-ring such that* $2R \neq \{0\}$ *with a* (σ,τ)*-derivation d such that* σ *commutes with* τ*,* σ *is an automorphism,* τ *is an epimorphism and* σ,τ *commute with d. If* d^2 *is a* (σ^2,τ^2)*-derivation, then* $d = 0$.

The conditions $2R = \{0\}$ in Theorem 3.1 and $2R \neq \{0\}$ in Corollary 3.1 are essential as the following example shows.

Example 3.2. Let $R = \mathbb{Z}_2[x]$. Then R is an integral domain which means that R is a commutative prime ring. Also, we have $2R = \{0\}$. If we take d to be the usual derivative on $R = \mathbb{Z}_2[x]$, then d is a $(1_R,1_R)$-derivation on R which is non-zero. But d^2 is also a $(1_R,1_R)$-derivation on $R = \mathbb{Z}_2[x]$ by Lemma 2.7.

The following result generalizes [12, Proposition 4.8].

Proposition 3.1. *Let R be a near-ring with a* (σ,τ)*-derivation d and an* (α,β)*-derivation D such that* α *commutes with* β*,* α *is an automorphism,* σ,β,τ *are epimorphisms and* α,β,τ *commute with D. If dD is a* $(\sigma\alpha,\tau\beta)$*-derivation and there exist* $x_o,y_o \in R$ *such that* $d(x_o),D(y_o)$ *are not left zero divisors in R, then* $2R = \{0\}$.

Proof. By the same way of the proof of Theorem 3.1, we will deduce that $d(R)D(R)(2D(R)) = \{0\}$. Since $d(x_o)$ is not a left zero divisor in R, we have $D(R)(2D(R)) = \{0\}$. Again, as $D(y_o)$ is not a left zero divisor in R, so $2D(R) = \{0\}$ which implies that $2R = \{0\}$ by Lemma 3.2(ii). ∎

4. Posner's second theorem

In this section we generalized Posner's second theorem for (σ,τ)-derivations.

Lemma 4.1. *Let R be a near-ring with a multiplicative epimorphism* θ*. If U is a non-zero semigroup right (left) ideal of R, then* $\theta(U)$ *is a semigroup right (left) ideal of R. Moreover, if* θ *is a multiplicative automorphism on R then* $\theta(U)$ *is a non-zero semigroup right (left) ideal of R.*

Proof. Let U be a non-zero semigroup right ideal of R and $x \in R$. Since θ is onto, there exists $r \in R$ such that $\theta(r) = x$. Thus, $\theta(u)x = \theta(u)\theta(r) = \theta(ur) \in \theta(U)$ for all $u \in U$. Hence, $\theta(U)$ is a semigroup right ideal of R. If θ is a multiplicative automorphism, then $\theta(U) = \{0\}$ implies $U = \{0\}$, a contradiction. The proof is similar for semigroup left ideals. ∎

The following result generalizes [2, Theorem 1] and [18, Lemma 3].

Theorem 4.1. *Let R be a prime ring with a non-zero* (σ,τ)*-derivation d such that* σ *or* τ *is an automorphism and* $\sigma(x)d(x) = d(x)\tau(x)$ *for all* $x \in U$*, where U is a non-zero semigroup ideal of R which is closed under addition. Then R is a commutative ring.*

Proof. Suppose τ is an automorphism. U is closed under addition implies $\sigma(x+y)d(x+y)=d(x+y)\tau(x+y)$ for all $x,y\in U$. So $\sigma(x)d(x)+\sigma(x)d(y)+\sigma(y)d(x)+\sigma(y)d(y)=d(x)\tau(x)+d(x)\tau(y)+d(y)\tau(x)+d(y)\tau(y)$. Using $\sigma(x)d(x)=d(x)\tau(x)$ and $\sigma(y)d(y)=d(y)\tau(y)$, we get

$$\sigma(x)d(y)+\sigma(y)d(x)=d(x)\tau(y)+d(y)\tau(x)\quad\text{for all}\quad x,y\in U. \tag{4.1}$$

Adding $d(x)\tau(y)+\sigma(y)d(x)$ to both sides of (4.1), we have $\sigma(x)d(y)+d(x)\tau(y)+2\sigma(y)d(x)=\sigma(y)d(x)+d(y)\tau(x)+2d(x)\tau(y)$ which means $d(xy)+2\sigma(y)d(x)=d(yx)+2d(x)\tau(y)$ and then for all $x,y\in U$, we get

$$d(xy)-d(yx)=2d(x)\tau(y)-2\sigma(y)d(x)=2(d(x)\tau(y)-\sigma(y)d(x)). \tag{4.2}$$

Replacing y by xy in (4.2) and using $\sigma(x)d(x)=d(x)\tau(x)$ for all $x\in U$, we have

$$\begin{aligned} d(xxy)-d(xyx)&=2(d(x)\tau(x)\tau(y)-\sigma(x)\sigma(y)d(x))\\ &=2(\sigma(x)d(x)\tau(y)-\sigma(x)\sigma(y)d(x))\\ &=\sigma(x)(2(d(x)\tau(y)-\sigma(y)d(x)))=\sigma(x)(d(xy)-d(yx)), \end{aligned}$$

On the other hand, we have

$$d(xxy)-d(xyx)=d(x(xy-yx))=\sigma(x)(d(xy)-d(yx))+d(x)\tau(xy-yx).$$

Comparing the last equations, we obtain $d(x)\tau(xy-yx)=0$, for all $x,y\in U$. Thus, we have the following

$$d(x)\tau(x)\tau(y)=d(x)\tau(y)\tau(x)\quad\text{for all}\quad x,y\in U. \tag{4.3}$$

Replacing y by yz and using (4.3), we get $d(x)\tau(y)\tau(x)\tau(z)=d(x)\tau(x)\tau(y)\tau(z)=d(x)\tau(y)\tau(z)\tau(x)$ for all $x,y,z\in U$. So $d(x)\tau(y)(\tau(x)\tau(z)-\tau(z)\tau(x))=0$. Thus, $d(x)\tau(U)(\tau(x)\tau(z)-\tau(z)\tau(x))=\{0\}$ for all $x,z\in U$. Using Lemma 4.1 and Lemma 2.11, we have for all $x\in U$ either $d(x)=0$ or $\tau(x)\tau(z)-\tau(z)\tau(x)=\tau(xz-zx)=0$ for all $z\in U$. If $d(U)=\{0\}$, then $d=0$ by Lemma 2.9(ii), a contradiction. So there exists $a\in U$ such that $d(a)\neq 0$ and hence $\tau(az-za)=0$ for all $z\in U$. But τ is an automorphism implies that $az-za=0$ for all $z\in U$ and then a centralizes U. Therefore, $a\in Z(R)$ by Lemma 2.8. Replacing y by ay in (4.2), we get $d(xay)-d(ayx)=2(d(x)\tau(a)\tau(y)-\sigma(a)\sigma(y)d(x))$ for all $x,y\in U$. But from (4.1), we have $\sigma(x)d(a)+\sigma(a)d(x)-d(a)\tau(x)=d(x)\tau(a)$. Substituting this in the last equation and using (4.2) and $a\in Z(R)$, it will be

$$\begin{aligned} d(xay)-d(ayx)&=2(\sigma(a)d(x)\tau(y)+(\sigma(x)d(a)-d(a)\tau(x))\tau(y)-\sigma(a)\sigma(y)d(x))\\ &=2\sigma(a)(d(x)\tau(y)-\sigma(y)d(x))+2((\sigma(x)d(a)-d(a)\tau(x))\tau(y))\\ &=\sigma(a)2(d(x)\tau(y)-\sigma(y)d(x))+2(\sigma(x)d(a)-d(a)\tau(x))\tau(y)\\ &=\sigma(a)(d(xy)-d(yx))-(d(ax)-d(xa))\tau(y)\\ &=\sigma(a)(d(xy)-d(yx)) \end{aligned}$$

for all $x,y\in U$ since $d(ax)-d(xa)=0$ for all $x\in U$. On the other hand, $d(xay)-d(ayx)=d(a(xy-yx))=\sigma(a)(d(xy)-d(yx))+d(a)\tau(xy-yx)$ for all $x,y\in U$. Comparing the last two equations, we get $d(a)\tau(xy-yx)=0$ and then $d(a)\tau(x)\tau(y)=d(a)\tau(y)\tau(x)$ for all $x,y\in U$. Putting xz instead of x where $z\in U$, we get $d(a)\tau(x)\tau(z)\tau(y)=d(a)\tau(y)\tau(x)\tau(z)=d(a)\tau(x)\tau(y)\tau(z)$ for all $x,y,z\in U$. Therefore, $d(a)\tau(x)(\tau(z)\tau(y)-\tau(y)\tau(z))=0$ for all $x,y,z\in U$. Thus, $d(a)\tau(U)(\tau(z)\tau(y)-\tau(y)\tau(z))=\{0\}$. Using $d(a)\neq 0$, Lemma 4.1 and Lemma 2.11, we have $\tau(z)\tau(y)-\tau(y)\tau(z)=\tau(zy-yz)=0=\tau(0)$ and then $zy=yz$ for all

$y,z \in U$. By Lemma 2.8, we obtain $U \subseteq Z(R)$. Hence, R is a commutative ring by Lemma 2.10. The proof for σ is an automorphism is similar. ∎

It is not true to replace the condition "$\sigma(x)d(x) = d(x)\tau(x)$" in Theorem 4.1 by "$xd(x) = d(x)x$" as the following example shows.

Example 4.1. Let R be the prime ring $M_2(\mathbb{Z}_2)$. Take $d = \tau$ is the identity map on R and $\sigma = 0$ (or $d = \sigma$ is the identity map on R and $\tau = 0$). Then d is a non-zero (σ,τ)-derivation d on R. Clearly that $d(x)x = xd(x) = x^2$ for all $x \in R$. But R is not commutative.

Corollary 4.1. *Let R be a prime ring with a non-zero σ-derivation d such that $\sigma(x)d(x) = d(x)x$ for all $x \in U$ where U is a non-zero semigroup ideal of R which is closed under addition. Then R is a commutative ring.*

Lemma 4.2. *Let R be an abelian near-ring with a non-zero (σ,τ)-derivation d such that σ and τ are epimorphisms. Then $d(\operatorname{dist}(R)) \subseteq \operatorname{dist}(R)$, where $\operatorname{dist}(R)$ is the set of distributive elements of R.*

Proof. For all $x,y \in R, s \in \operatorname{dist}(R)$, we have $d((x+y)s) = d(xs+ys)$. That means $\sigma(x+y)d(s)+d(x+y)\tau(s) = \sigma(x)d(s)+d(x)\tau(s)+\sigma(y)d(s)+d(y)\tau(s)$. Since τ is onto, we get $\tau(s) \in \operatorname{dist}(R)$. It follows that $(\sigma(x)+\sigma(y))d(s)+d(x)\tau(s)+d(y)\tau(s) = \sigma(x)d(s)+\sigma(y)d(s)+d(x)\tau(s)+d(y)\tau(s)$ and hence $(\sigma(x)+\sigma(y))d(s) = \sigma(x)d(s)+\sigma(y)d(s)$. So $d(s) \in \operatorname{dist}(R)$. ∎

Theorem 4.2. *Let R be an integral near-ring with a non-zero (σ,τ)-derivation d such that σ and τ are automorphisms and $\sigma(x)d(x) = d(x)\tau(x)$ for all $x \in R$. Then d is a (σ,σ)-derivation on $\operatorname{dist}(R)$ and either $d(\operatorname{dist}(R)) = 0$ or $\operatorname{dist}(R)$ is a commutative ring. Moreover, if $d(\operatorname{dist}(R)) \neq 0$, then $\sigma(s) = \tau(s)$ for all $s \in \operatorname{dist}(R)$.*

Proof. For all $x,y \in R$, we have $d(x(x+y)) = d(x^2+xy)$. So

$$\begin{aligned} d(x(x+y)) &= \sigma(x)d(x+y)+d(x)\tau(x+y) \\ &= \sigma(x)d(x)+\sigma(x)d(y)+d(x)\tau(x)+d(x)\tau(y) \\ &= \sigma(x)d(x)+\sigma(x)d(y)+\sigma(x)d(x)+d(x)\tau(y) \end{aligned}$$

as $d(x)\tau(x) = \sigma(x)d(x)$. On the other hand

$$\begin{aligned} d(x^2+xy) = d(x^2)+d(xy) &= \sigma(x)d(x)+d(x)\tau(x)+\sigma(x)d(y)+d(x)\tau(y) \\ &= \sigma(x)d(x)+\sigma(x)d(x)+\sigma(x)d(y)+d(x)\tau(y). \end{aligned}$$

After cancellation we get $\sigma(x)d(y)+\sigma(x)d(x) = \sigma(x)d(x)+\sigma(x)d(y)$ for all $x,y \in R$. Thus, $0 = \sigma(x)(d(y)+d(x)-d(y)-d(x)) = \sigma(x)d(y+x-y-x)$ for all $x,y \in R$. Since R is without zero divisors and σ is an automorphism, either $x = 0$ or $d(y+x-y-x) = 0$ for all $0 \neq x \in R$ and for all $y \in R$. But if $x = 0$, then $d(y+x-y-x) = d(y-y) = d(0) = 0$. So $d((x,y)) = 0$ for all $x,y \in R$. Since $z(x,y) = (zx,zy)$ for all $x,y,z \in R$, we have $d(z(x,y)) = 0$ and then $0 = d(z(x,y)) = \sigma(z)d((x,y))+d(z)\tau(x,y) = d(z)\tau(x,y)$. Since $d \neq 0$, there exists $z \in R$ such that $d(z) \neq 0$ and then $\tau(x,y) = 0$ for all $x,y \in R$. It follows that $(R,+)$ is an abelian group. So R is an abelian near-ring. Thus, $\operatorname{dist}(R)$ is a subnear-ring of R which is an integral ring. Also, $d(\operatorname{dist}(R)) \subseteq \operatorname{dist}(R)$ by Lemma 4.2. Therefore, $d(\operatorname{dist}(R)) = 0$ or $\operatorname{dist}(R)$ is a commutative ring by Theorem 4.1. Now, If $d(\operatorname{dist}(R)) = 0$, then d is a (σ,σ)-derivation on $\operatorname{dist}(R)$. Suppose that $d(\operatorname{dist}(R)) \neq 0$. So $\sigma(s)d(s) = d(s)\tau(s)$ for all $s \in \operatorname{dist}(R)$. Thus, $d(s)(\sigma(s)-\tau(s)) = 0$ and either $d(s) = 0$ or $\sigma(s) = \tau(s)$. That means if

$d(s) \neq 0$, then $\sigma(s) = \tau(s)$. Since $d(\text{dist}(R)) \neq 0$, there exists $t \in \text{dist}(R)$ such that $d(t) \neq 0$. So for all $s \in \text{dist}(R) - \{0\}$ such that $d(s) = 0$, we get $\sigma(ts)d(ts) = d(ts)\tau(ts)$. It follows that $\sigma(t)\sigma(s)d(t)\tau(s) = d(t)\tau(s)\tau(t)\tau(s)$. As $\text{dist}(R)$ is a commutative integral ring, τ is an automorphism and $\sigma(t) = \tau(t)$ where $d(t) \neq 0$ and $t \in \text{dist}(R)$, we have $\sigma(s) = \tau(s)$ for all $s \in \text{dist}(R)$. Also, σ is an automorphism on R implies that σ is an automorphism on $\text{dist}(R)$. Therefore, d is a non-zero (σ,σ)-derivation on $\text{dist}(R)$. ∎

The following result generalizes [1, Theorem 1].

Theorem 4.3. *Let R be a prime ring with a non-zero (σ,τ)-derivation d such that σ,τ are epimorphisms and $\sigma(x)d(x) = d(x)\tau(x)$ for all $x \in U$ where U is a non-zero right (left) ideal of R. Then $\tau(U) = \{0\}$ or $\sigma(U) = \{0\}$ or (R is a commutative ring and $\sigma = \tau$).*

Proof. Suppose U is a non-zero right ideal. The first part of the proof is similar to the first part of the proof of Theorem 4.1 up to equation (4.3)

$$d(x)\tau(x)\tau(y) = d(x)\tau(y)\tau(x) \quad \text{for all} \quad x,y \in U.$$

Replacing y by yz and using (4.3), we have $d(x)\tau(y)\tau(x)\tau(z) = d(x)\tau(x)\tau(y)\tau(z) = d(x)\tau(y)\tau(z)\tau(x)$ for all $x,y,z \in U$, which means $d(x)\tau(y)(\tau(x)\tau(z) - \tau(z)\tau(x)) = 0$. Thus, $d(x)\tau(U)(\tau(x)\tau(z) - \tau(z)\tau(x)) = \{0\}$ for all $x,z \in U$. By Lemma 4.1, either $\tau(U) = \{0\}$ or $d(x)\tau(U)R(\tau(x)\tau(z) - \tau(z)\tau(x)) = \{0\}$. If $\tau(U) \neq \{0\}$, then for each $x \in U$ either $d(x)\tau(U) = \{0\}$ or $\tau(xz) = \tau(zx)$ for all $z \in U$. Let $A = \{x \in U : d(x)\tau(U) = \{0\}\}$ and $B = \{x \in U : \tau(xz) = \tau(zx)$ for all $z \in U\}$. Then A and B are subgroups of $(U,+)$ and $A \cup B = U$. Thus, $A = U$ or $B = U$. In other words, $d(U)\tau(U) = \{0\}$ or $\tau(U) \subseteq Z(R)$. Suppose $d(U)\tau(U) = \{0\}$. So (4.1) will be $\sigma(x)d(y) + \sigma(y)d(x) = 0$ for all $x,y \in U$. Since $d(xy) = \sigma(x)d(y), d(yx) = \sigma(y)d(x)$, we have

$$d(xy + yx) = 0 \quad \text{for all} \quad x,y \in U. \tag{4.4}$$

Replacing x,y by $z,(xy+yx)$ respectively in (4.4), we get $d(z(xy+yx) + (xy+yx)z) = 0$ for all $x,y,z \in U$. It follows that

$$0 = \sigma(z)d(xy+yx) + d(z)\tau(xy+yx) + \sigma(xy+yx)d(z) + d(xy+yx)\tau(z) \tag{4.5}$$

for all $x,y,z \in U$. Observe that $d(xy+yx)\tau(z) = d(z)\tau(xy+yx) = 0$ from $d(U)\tau(U) = \{0\}$ and $\sigma(z)d(xy+yx) = 0$ from (4.4). Thus, (4.5) will be $\sigma(xy+yx)d(z) = 0$. Replacing y by yz, it yields $0 = \sigma(xyz+yzx)d(z) = \sigma(x)\sigma(y)\sigma(z)d(z) + \sigma(y)\sigma(z)\sigma(x)d(z) = \sigma(y)\sigma(z)\sigma(x)d(z)$ for all $x,y,z \in U$ since $\sigma(z)d(z) = d(z)\tau(z) = 0$. Replacing y by yr where $r \in R$, we get $\sigma(y)\sigma(r)\sigma(z)\sigma(x)d(z) = 0$. As R is prime and σ is onto, either $\sigma(U) = \{0\}$ or $\sigma(z)\sigma(x)d(z) = 0$ for all $x,z \in U$. If $\sigma(U) \neq \{0\}$, then $\sigma(z)\sigma(x)d(z) = 0$ for all $x,z \in U$. Putting xr instead of x, we conclude that $\sigma(z)\sigma(x)Rd(z) = \{0\}$ and then for every $z \in U$ either $d(z) = 0$ or $\sigma(z)\sigma(x) = \sigma(zx) = 0$. Let $A = \{u \in U : d(u) = 0\}$ and $B = \{u \in U : \sigma(ux) = 0$ for all $x \in U\}$. So A and B are subgroups of $(U,+)$. Moreover, $U = A \cup B$. Thus, either $A = U$ or $B = U$. If $A = U$, then $d(U) = \{0\}$ and hence $d = 0$ by Lemma 2.9(i), a contradiction with the hypothesis. If $B = U$, then $\sigma(U^2) = \{0\}$ which implies $\sigma(U)\sigma(U) = \{0\}$. But $\sigma(U)$ is a non-zero semigroup right ideal of R by Lemma 4.1 and $\sigma(U) \neq \{0\}$. So $\sigma(U)\sigma(U) \neq \{0\}$, a contradiction. Hence, $d(U)\tau(U) \neq \{0\}$ if $\sigma(U) \neq \{0\}$. Therefore, $\tau(U) \subseteq Z(R)$. But $\tau(U) \neq \{0\}$ is a non-zero semigroup right ideal of R, so R is a commutative ring by Lemma 2.10. It follows that $\sigma(x)d(x) = d(x)\tau(x)$ implies $d(x)(\sigma(x) - \tau(x)) = 0$ for all $x \in U$. Since R is a commutative prime ring, it doesn't have non-zero zero divisors by Lemma 2.4. Thus, either $d(x) = 0$ or $\sigma(x) = \tau(x)$.

Let $A = \{x \in U | d(x) = 0\}$ and $B = \{x \in U | \sigma(x) = \tau(x)\}$. Then A and B are subgroups of U whose union is U. As $d(U) \neq 0$, we have $B = U$ and $\sigma(x) = \tau(x)$ for all $x \in U$. Hence, $\sigma(ux) = \tau(ux)$ for all $u \in U$ and $x \in R$. That implies $\sigma(u)(\sigma(x) - \tau(x)) = 0$. Since $\sigma(U) \neq \{0\}$, we get $\sigma(x) = \tau(x)$ for all $x \in R$ and $\sigma = \tau$. The proof when U is a non-zero left ideal is similar. ∎

If a 3-prime near-ring R with a (σ,σ)-derivation d such that $\sigma(x)d(x) = d(x)\sigma(x)$ for all $x \in R$, then R need not be a ring as the following example shows:

Example 4.2. Let $R = I \times I$ as a set, where I is any integral ring with identity which has at least three elements. Define the addition and the multiplication on R by $(a,b)+(c,d) = (a+c, b+d)$ and $(a,b)(c,d) = (ac, bc+d)$ if $(a,b) \neq (0,0)$ and $(0,0)(c,d) = (0,0)$. Then R is a zero-symmetric abelian near-ring with identity $(1,0)$ which is not a ring. Let D be a non-zero derivation on I and σ the endomorphism defined on R by $\sigma((a,b)) = (a,0)$ for all $(a,b) \in R$. Define $d : R \to R$ by $d((a,b)) = (D(a),0)$. Then d is a non-zero (σ,σ)-derivation on R by simple calculations.

Observe that R is 3-prime. Indeed, assume that $(a,b)R(c,d) = (0,0)$ with $(a,b) \neq (0,0)$. If $a \neq 0$, then $(a,b)(1,0)(c,d) = (0,0)$. That means $(a,b)(c,d) = (ac, bc+d) = (0,0)$. Thus, $c = 0$ and hence $d = 0$. Now, suppose $a = 0$ and $b \neq 0$. It follows that $(0,0) = (0,b)(0,1)(c,d) = (0,1)(c,d) = (0,c+d)$ and then $c = -d$. It follows that $(0,0) = (0,b)(0,y)(-d,d) = (0,y)(-d,d) = (0,-yd+d) = (0,(-y+1)d)$ for all $y \in I - \{0\}$. If $d \neq 0$, then $y = 1$ and $I = \{0,1\}$ which is a contradiction with the number of elements of I. Therefore, $d = 0$ and $(c,d) = (0,0)$. Hence, R is a 3-prime near-ring.

Now, choose I to be the integral domain $\mathbb{R}[x]$ where $\mathbb{R}$ is the field of real numbers and choose D to be usual derivative on $\mathbb{R}[x]$. Observe that we have $\sigma(a,b)d((a,b)) = d((a,b))\sigma(a,b)$ for all $(a,b) \in R$, but R is not a ring.

Proposition 4.1. *Let R be a prime ring.*

(i) *If $nx = 0$ for some $x \in R$ and a positive integer n, then either $nR = \{0\}$ or $x = 0$.*
(ii) *If $nR \neq \{0\}$ for some positive integer n and $nx \in Z(R)$ for some $x \in R$, then $x \in Z(R)$.*

Proof. (i) For all $y,z \in R$, we have $0 = yz(nx) = n(yzx) = (ny)zx$. From the primeness of R, we have either $nR = \{0\}$ or $x = 0$.

(ii) If $Z(R) = \{0\}$, then $nx = 0$ and hence $x = 0$ by using (i). If $Z(R) \neq \{0\}$, then there exists $z \in Z(R) - \{0\}$. Observe that $ny \neq 0$ for all $y \in R - \{0\}$ from (i). Now, $z(nx) \in Z(R)$. Observe that $z(nx) = n(zx) = (nz)x \in Z(R)$. But $nz \in Z(R) - \{0\}$. Therefore, $x \in Z(R)$ by Lemma 2.3. ∎

The following example shows that the hypothesis "prime ring" in Proposition 4.1 can't be replaced by "3-prime near-ring".

Example 4.3. Let $R = M_o(G)$, where G is the abelian group $(\mathbb{Z}_4, +)$. Then $M_o(G)$ is 3-prime. Take $f \in M_o(G)$ such that $xf = 2x$ for all $x \in G$. Then $2f = 0$, but neither $2M_o(G) = \{0\}$ nor $f = 0$. Observe that $2f \in Z(M_o(G))$ and $2M_o(G) \neq \{0\}$, but $f \notin Z(M_o(G))$ since $fg \neq gf$, where $g \in M_o(G)$ is defined by $\{0,1,3\}g = \{0\}$ and $2g = 1$.

Lemma 4.3. *Let R be a ring and σ and τ are endomorphisms of R. Then for all $x,y,z \in R$, we have the following relations:*

(i) $[x, y \pm z]_{\sigma,\tau} = [x,y]_{\sigma,\tau} \pm [x,z]_{\sigma,\tau}$.
(ii) $[x \pm y, z]_{\sigma,\tau} = [x,z]_{\sigma,\tau} \pm [y,z]_{\sigma,\tau}$.

(iii) $[xy,z]_{\sigma,\tau} = \sigma(x)[y,z]_{\sigma,\tau} + [x,z]_{\sigma,\tau}\tau(y)$.
(iv) $[x,yz]_{\sigma,\tau} = y[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\sigma}z$.

Proof. (i) For all $x,y,z \in R$, we have $[x,y \pm z]_{\sigma,\tau} = \sigma(x)(y \pm z) - (y \pm z)\tau(x) = \sigma(x)y \pm \sigma(x)z - y\tau(x) \pm (-z\tau(x)) = \sigma(x)y - y\tau(x) \pm (\sigma(x)z - z\tau(x)) = [x,y]_{\sigma,\tau} \pm [x,z]_{\sigma,\tau}$.

(ii) For all $x,y,z \in R$, we have $[x \pm y,z]_{\sigma,\tau} = \sigma(x \pm y)z - z\tau(x \pm y) = \sigma(x)z \pm \sigma(y)z - z\tau(x) \pm (-z\tau(y)) = \sigma(x)z - z\tau(x) \pm (\sigma(y)z - z\tau(y)) = [x,z]_{\sigma,\tau} \pm [y,z]_{\sigma,\tau}$.

(iii) For all $x,y,z \in R$, we have $[xy,z]_{\sigma,\tau} = \sigma(xy)z - z\tau(xy) = \sigma(x)\sigma(y)z - z\tau(x)\tau(y) = \sigma(x)\sigma(y)z + (-\sigma(x)z\tau(y) + \sigma(x)z\tau(y)) - z\ \tau(x)\tau(y) = \sigma(x)(\sigma(y)z - z\tau(y)) + (\sigma(x)z - z\tau(x))\tau(y) = \sigma(x)[y,z]_{\sigma,\tau} + [x,z]_{\sigma,\tau}\tau(y)$.

(iv) For all $x,y,z \in R$, we have $[x,yz]_{\sigma,\tau} = \sigma(x)yz - yz\tau(x) = \sigma(x)yz + (-y\sigma(x)z + y\sigma(x)z) - yz\tau(x) = (\sigma(x)y - y\sigma(x))z + y(\sigma(x)z - z\tau(x)) = [x,y]_{\sigma,\sigma}z + y[x,z]_{\sigma,\tau} = y[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\sigma}z$. ∎

It is not true in general that $[x,yz]_{\sigma,\tau} = y[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\tau}z$ as the following example shows.

Example 4.4. Let R be a ring. Choose $\sigma = 1_R$ and $\tau = 0$. Then for all $x,y,z \in R$, we have $[x,yz]_{\sigma,\tau} = \sigma(x)yz - yz\tau(x) = xyz$ and $y[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\tau}z = y(\sigma(x)z - z\tau(x)) + (\sigma(x)y - y\tau(x))z = yxz + xyz$.

Lemma 4.4. *Let R be a ring with (σ,τ)-derivations d and D. Then*

(i) *[13, Example 3.1] $\delta : R \to R$ such that $\delta(x) = \sigma(x)a - a\tau(x)$ for all $x \in R$ is a (σ,τ)-derivation on R for all $a \in R$.*
(ii) *$g : R \to R$ such that $g(x) = ad(x)$ for all $x \in R$ is a (σ,τ)-derivation on R, where $a \in Z(R)$.*
(iii) *$d + D$ is a (σ,τ)-derivation on R.*

Proof. (ii) For all $x,y \in R$, we have $g(x+y) = ad(x+y) = a(d(x)+d(y)) = ad(x)+ad(y) = g(x)+g(y)$. Also, $g(xy) = ad(xy) = a(\sigma(x)d(y)+d(x)\tau(y)) = \sigma(x)ad(y)+ad(x)\tau(y) = \sigma(x)g(y)+g(x)\tau(y)$.

(iii) Clearly that $d+D$ is additive mapping. Now,

$$\begin{aligned}(d+D)(xy) &= d(xy)+D(xy) = \sigma(x)d(y)+d(x)\tau(y)+\sigma(x)D(y)+D(x)\tau(y)\\ &= \sigma(x)(d(y)+D(y))+(d(x)+D(x))\tau(y)\\ &= \sigma(x)(d+D)(y)+(d+D)(x)\tau(y).\end{aligned}$$

Therefore, $d+D$ is also a (σ,τ)-derivation on R. ∎

Theorem 4.4. *Let R be a prime ring with a non-zero (σ,τ)-derivation d, σ and τ are epimorphisms of R. If $\sigma(x)d(x) - d(x)\tau(x) \in Z(R)$, for all $x \in R$, then R is a commutative ring or $d(Z(R)) = \{0\}$.*

Proof. Observe that $\sigma(x)d(x) - d(x)\tau(x) = [x,d(x)]_{\sigma,\tau}$ for all $x \in R$. From $[x+y,d(x+y)]_{\sigma,\tau} \in Z(R)$ for all $x,y \in R$ and using Lemma 4.3, we have $[x,d(x)]_{\sigma,\tau} + [x,d(y)]_{\sigma,\tau} + [y,d(x)]_{\sigma,\tau} + [y,d(y)]_{\sigma,\tau} \in Z(R)$. Using $[x,d(x)]_{\sigma,\tau} \in Z(R)$, $[y,d(y)]_{\sigma,\tau} \in Z(R)$ and that $Z(R)$ is a subring of R, we get

$$[x,d(y)]_{\sigma,\tau} + [y,d(x)]_{\sigma,\tau} \in Z(R) \quad \text{for all} \quad x,y \in R. \tag{4.6}$$

If $Z(R) = \{0\}$, then $\sigma(x)d(x) - d(x)\tau(x) = 0$ for all $x \in R$ and hence R is a commutative ring by Theorem 4.3. So $R = Z(R) = \{0\}$ and $d = 0$, a contradiction. Therefore, $Z(R) \neq \{0\}$. We divide the proof into two cases:

(i) R is not of characteristic 2. Then there exists $c \in Z(R) - \{0\}$ such that $[x, d(c)]_{\sigma,\tau} + [c, d(x)]_{\sigma,\tau} \in Z(R)$ for all $x \in R$ by (4.6). Write $d_1(x) = [x, d(c)]_{\sigma,\tau}$ and $d_2(x) = [c, d(x)]_{\sigma,\tau}$. Observe that d_1, d_2 and $d_1 + d_2$ are (σ, τ)-derivations by Lemma 4.4. If $d_1 + d_2 \neq 0$, then $(d_1 + d_2)(R) \subseteq Z(R)$ implies that R is a commutative ring by Lemma 2.12. If $d_1 + d_2 = 0$, then $[x, d(c)]_{\sigma,\tau} + [c, d(x)]_{\sigma,\tau} = 0$ for all $x \in R, c \in Z(R)$. It follows that $0 = [c, d(c)]_{\sigma,\tau} + [c, d(c)]_{\sigma,\tau} = 2[c, d(c)]_{\sigma,\tau}$ and hence $[c, d(c)]_{\sigma,\tau} = 0$ by Proposition 4.1(i). As $\sigma(c), \tau(c) \in Z(R)$, we obtain $[c, d(c)]_{\sigma,\tau} = d(c)(\sigma(c) - \tau(c)) = 0$. Thus, for all $c \in Z(R)$, either $d(c) = 0$ or $\sigma(c) = \tau(c)$. If $\sigma(c) \neq \tau(c)$ and $d(c) = 0$ for some $c \in Z(R)$, then $d_1 = 0$ which implies $d_2 = 0$. Thus, $(\sigma(c) - \tau(c))d(x) = 0$ for all $x \in R$ and $d = 0$ by Lemma 2.4, a contradiction. So if $d(Z(R)) = \{0\}$, then $\sigma(a) = \tau(a)$ for all $a \in Z(R)$. If $d(c) \neq 0$ and $\sigma(c) = \tau(c)$ for some $c \in Z(R)$, then $d_2 = 0$. So $d_1(x) = \sigma(x)d(c) - d(c)\tau(x) = 0$ for all $x \in R$. If there exists $a \in Z(R)$ such that $\sigma(a) \neq \tau(a)$, then $d(c)(\sigma(a) - \tau(a)) = 0$ and $d(c) = 0$, a contradiction. So if $d(c) \neq 0$ for some $c \in Z(R)$, then $\sigma(a) = \tau(a)$ for all $a \in Z(R)$. Now, we have the following case: $d_1 = d_2 = 0$, $d(Z(R)) \neq \{0\}$ and $\sigma(a) = \tau(a)$ for all $a \in Z(R)$. Replacing y in (4.6) by zy and using Lemma 4.3(i), (iii) and (iv), we get for all $x, y, z \in R$

$$\begin{aligned}
&[x, d(zy)]_{\sigma,\tau} + [zy, d(x)]_{\sigma,\tau} \\
&= [x, \sigma(z)d(y) + d(z)\tau(y)]_{\sigma,\tau} + [zy, d(x)]_{\sigma,\tau} \\
&= [x, \sigma(z)d(y)]_{\sigma,\tau} + [x, d(z)\tau(y)]_{\sigma,\tau} + \sigma(z)[y, d(x)]_{\sigma,\tau} + [z, d(x)]_{\sigma,\tau}\tau(y) \\
&= \sigma(z)[x, d(y)]_{\sigma,\tau} + [x, \sigma(z)]_{\sigma,\sigma}d(y) + d(z)[x, \tau(y)]_{\sigma,\tau} + [x, d(z)]_{\sigma,\sigma}\tau(y) \\
&\quad + \sigma(z)[y, d(x)]_{\sigma,\tau} + [z, d(x)]_{\sigma,\tau}\tau(y) \\
&= \sigma(z)([x, d(y)]_{\sigma,\tau} + [y, d(x)]_{\sigma,\tau}) + ([x, d(z)]_{\sigma,\sigma} + [z, d(x)]_{\sigma,\tau})\tau(y) \\
&\quad + [x, \sigma(z)]_{\sigma,\sigma}d(y) + d(z)[x, \tau(y)]_{\sigma,\tau}.
\end{aligned}$$

Putting $z = c \in Z(R)$, using $d_2 = 0$ and (4.6), we deduce that $[x, d(c)]_{\sigma,\sigma}\tau(y) + d(c)[x, \tau(y)]_{\sigma,\tau} \in Z(R)$ for all $x, y \in R$. Then $\sigma(x)d(c)\tau(y) - d(c)\sigma(x)\tau(y) + d(c)\sigma(x)\tau(y) - d(c)\tau(y)\tau(x) = \sigma(x)d(c)\tau(y) - d(c)\tau(y)\tau(x) \in Z(R)$ for all $x, y \in R$. Suppose $d(c) \neq 0$ for some $c \in Z(R)$ and assume that

$$\sigma(x)d(c)\tau(y) = d(c)\tau(y)\tau(x) \quad \text{for all} \quad x, y \in R. \tag{4.7}$$

Multiplying both sides by $\tau(z)$ from the right, we obtain

$$\sigma(x)d(c)\tau(y)\tau(z) = d(c)\tau(y)\tau(x)\tau(z) \quad \text{for all} \quad x, y, z \in R. \tag{4.8}$$

Replacing y by yz in (4.7), we have

$$\sigma(x)d(c)\tau(y)\tau(z) = d(c)\tau(y)\tau(z)\tau(x) \quad \text{for all} \quad x, y, z \in R. \tag{4.9}$$

From (4.8) and (4.9), we conclude $d(c)\tau(y)(\tau(z)\tau(x) - \tau(x)\tau(z)) = 0$ for all $x, y, z \in R$. Since R is prime and $d(c) \neq 0$, we obtain that R is commutative. Now, assume that $\tau(a) \neq 0$ for some $a \in R$ such that $\sigma(x)d(c)\tau(a) \neq d(c)\tau(a)\tau(x)$. It follows that $\delta(x) = \sigma(x)d(c)\tau(a) - d(c)\tau(a)\tau(x) \in Z(R)$ for all $x \in R$ is a non-zero inner (σ, τ)-derivation and R is a commutative ring by Lemma 2.12.

(ii) R is of characteristic 2. Adding $d(x)\tau(y) + d(y)\tau(x) - d(x)\tau(y) - d(y)\tau(x) = 0$ to (4.6), we have $\sigma(x)d(y) + d(x)\tau(y) - 2d(x)\tau(y) + \sigma(y)d(x) + d(y)\tau(x) - 2d(y)\tau(x) \in Z(R)$ which means

$$d(xy + yx) \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{4.10}$$

Now, suppose $d(Z(R)) \neq \{0\}$ and there exists $c \in Z(R) - \{0\}$ such that $d(c) \neq 0$. Replace y by yc in (4.10). Then $d(xyc + ycx) = d(c(xy + yx)) \in Z(R)$ for all $x, y \in R$. It follows that $\sigma(c)d(xy+yx) + d(c)\tau(xy+yx) \in Z(R)$. Since $\sigma(c)d(xy+yx) \in Z(R)$, we have $d(c)\tau(xy+yx) \in Z(R)$ and then $d(c)(uv+vu) \in Z(R)$ for all $u, v \in R$ as τ is onto. Firstly, suppose that $d(c)(xy+yx) = 0$ for all $x, y \in R$. So $d(c)xy = d(c)yx$ for all $x, y \in R$. Replacing x by xz in the last equation, we get $d(c)xzy = d(c)yxz = d(c)xyz$ and hence $d(c)x(zy - yz) = 0$ for all $x, y, z \in R$. The primeness of R and $d(c) \neq 0$ imply that R is commutative. Now, suppose $d(c)(st + ts) \in Z(R) - \{0\}$ for some $s, t \in R$. Using $d(c)(xy + yx) \in Z(R)$ for all $x, y \in R$ and replacing x by $[s,t]x$ and y by $[s,t]y$, we have $d(c)([s,t]x[s,t]y + [s,t]y[s,t]x) \in Z(R)$. Thus, $d(c)[s,t](x[s,t]y + y[s,t]x) \in Z(R)$. Since $d(c)[s,t] \in Z(R) - \{0\}$, it is not a zero divisor by Lemma 2.4. It follows that $(x[s,t]y + y[s,t]x) \in Z(R)$ for all $x, y \in R$. Replacing x by c and putting $a = [s,t]$, we obtain $c(ay + ya) \in Z(R)$. Again, by Lemma 2.3, we have $ay + ya \in Z(R)$ for all $y \in R$. Define $d_a : R \to R$ by $d_a(y) = ay + ya$ for all $y \in R$. Then d_a is an inner derivation on R and $d_a(R) \subseteq Z(R)$. If d_a is non-zero, then R is commutative by Lemma 2.12. If $d_a = 0$, then $a = [s,t] \in Z(R) - \{0\}$. Using Lemma 2.3, we get $d(c) \in Z(R) - \{0\}$. Thus, $d(c)(xy + yx) \in Z(R)$ for all $x, y \in R$ implies $xy + yx \in Z(R)$ for all $x, y \in R$. If there exists $b \in R$ such that $by + yb \neq 0$ for some $y \in R$, then d_b is a non-zero derivation on R and $d_b(R) \subseteq Z(R)$ which implies R to be a commutative ring by Lemma 2.12 and hence $by + yb = 0$, a contradiction. Thus, $xy + yx = 0$ and then R is a commutative ring. ∎

Corollary 4.2. *Let R be a prime ring of characteristic 2 with a non-zero (σ, τ)-derivation d such that σ and τ are automorphisms and commute with d. If $\sigma(x)d(x) + d(x)\tau(x) \in Z(R)$ for all $x \in R$., then R is a commutative ring or $d^2 = 0$.*

Proof. Using Theorem 4.4, R is a commutative ring or $d(Z(R)) = \{0\}$. If $d(Z(R)) = \{0\}$, then $d^2(xy) = d^2(yx)$ for all $x, y \in R$ from (4.10) in the proof of Theorem 4.4. Using Lemma 2.7, d^2 is a (σ^2, τ^2)-derivation on R. So by Lemma 2.13, R is a commutative ring or $d^2 = 0$. ∎

The following result generalizes Theorem 1 (in its part of derivations) of [14] and [8, Theorem 4].

Theorem 4.5. *Let R be a prime ring with a non-zero (σ, σ)-derivation d such that σ is an epimorphism and $\sigma(x)d(x) - d(x)\sigma(x) \in Z(R)$ for all $x \in U$, where U is a non-zero right (left) ideal of R. Then R is a commutative ring or $\sigma(U) = \{0\}$.*

Proof. From $[x+y, d(x+y)]_{\sigma,\sigma} \in Z(R)$ for all $x, y \in U$, we have

$$[x, d(y)]_{\sigma,\sigma} + [y, d(x)]_{\sigma,\sigma} \in Z(R) \text{ for all } x, y \in U. \tag{4.11}$$

We divide the proof into two cases:

(i) R is not of characteristic 2. Replacing y in (4.11) by x^2 and using Lemma 4.3, we get

$$\begin{aligned}
&[x, d(xx)]_{\sigma,\sigma} + [xx, d(x)]_{\sigma,\sigma} \\
&= [x, \sigma(x)d(x) + d(x)\sigma(x)]_{\sigma,\sigma} + [xx, d(x)]_{\sigma,\sigma} \\
&= [x, \sigma(x)d(x)]_{\sigma,\sigma} + [x, d(x)\sigma(x)]_{\sigma,\sigma} + \sigma(x)[x, d(x)]_{\sigma,\sigma} + [x, d(x)]_{\sigma,\sigma}\sigma(x) \\
&= \sigma(x)[x, d(x)]_{\sigma,\sigma} + [x, d(x)]_{\sigma,\sigma}\sigma(x) + 2\sigma(x)[x, d(x)]_{\sigma,\sigma} = 4\sigma(x)[x, d(x)]_{\sigma,\sigma}
\end{aligned}$$

and hence $4\sigma(x)[x, d(x)]_{\sigma,\sigma} \in Z(R)$. It follows that $\sigma(x)[x, d(x)]_{\sigma,\sigma} \in Z(R)$ by Proposition 4.1(ii). If $[x, d(x)]_{\sigma,\sigma} \neq 0$, then $\sigma(x) \in Z(R)$ by using Lemma 2.3. But that means

$[x,d(x)]_{\sigma,\sigma} = 0$, a contradiction. Thus, $[x,d(x)]_{\sigma,\sigma} = 0$ for all $x \in U$. Therefore, R is a commutative ring or $\sigma(U) = \{0\}$ by Theorem 4.3.

(ii) R is of characteristic 2. Using Lemma 4.3(ii), (iii) and $[x,d(x)]_{\sigma,\sigma} \in Z(R)$, we have for all $x,y \in U$

$$\begin{aligned}
&[xy+yx,d(x)]_{\sigma,\sigma} + [x^2,d(y)]_{\sigma,\sigma} \\
&= [xy,d(x)]_{\sigma,\sigma} + [yx,d(x)]_{\sigma,\sigma} + [x^2,d(y)]_{\sigma,\sigma} \\
&= \sigma(x)[y,d(x)]_{\sigma,\sigma} + [x,d(x)]_{\sigma,\sigma}\sigma(y) + \sigma(y)[x,d(x)]_{\sigma,\sigma} + [y,d(x)]_{\sigma,\sigma}\sigma(x) \\
&\quad + \sigma(x)[x,d(y)]_{\sigma,\sigma} + [x,d(y)]_{\sigma,\sigma}\sigma(x) \\
&= \sigma(x)[y,d(x)]_{\sigma,\sigma} + \sigma(x)[x,d(y)]_{\sigma,\sigma} + [y,d(x)]_{\sigma,\sigma}\sigma(x) + [x,d(y)]_{\sigma,\sigma}\sigma(x) \\
&= \sigma(x)([y,d(x)]_{\sigma,\sigma} + [x,d(y)]_{\sigma,\sigma}) + ([y,d(x)]_{\sigma,\sigma} + [x,d(y)]_{\sigma,\sigma})\sigma(x) = 0
\end{aligned}$$

using (4.11). So

$$[xy+yx,d(x)]_{\sigma,\sigma} + [x^2,d(y)]_{\sigma,\sigma} = 0 \quad \text{for all} \quad x,y \in U. \tag{4.12}$$

Using $d(x)\sigma(y) + d(y)\sigma(x) - d(x)\sigma(y) - d(y)\sigma(x) = 0$ for all $x,y \in U$ and (4.11), we have

$$\sigma(x)d(y) + d(x)\sigma(y) - 2d(x)\sigma(y) + \sigma(y)d(x) + d(y)\sigma(x) - 2d(y)\sigma(x) \in Z(R)$$

and consequently, we get

$$d(xy+yx) \in Z(R) \quad \text{for all} \quad x,y \in U. \tag{4.13}$$

Replacing y by $xy+yx$ in (4.12) and using (4.13), we have

$$\begin{aligned}
0 &= [x(xy+yx) + (xy+yx)x, d(x)]_{\sigma,\sigma} + [x^2, d(xy+yx)]_{\sigma,\sigma} \\
&= [xxy + xyx + xyx + yxx, d(x)]_{\sigma,\sigma} = [xxy + yxx, d(x)]_{\sigma,\sigma}.
\end{aligned}$$

Replacing y by xy in the last equation and using Lemma 4.3(iii), we get

$$\begin{aligned}
0 &= [xxxy + xyxx, d(x)]_{\sigma,\sigma} = [x(xxy+yxx), d(x)]_{\sigma,\sigma} \\
&= \sigma(x)[xxy+yxx, d(x)]_{\sigma,\sigma} + [x,d(x)]_{\sigma,\sigma}\sigma(xxy+yxx) \\
&= [x,d(x)]_{\sigma,\sigma}\sigma(xxy+yxx).
\end{aligned}$$

If there exists $a \in U$ such that $[a,d(a)]_{\sigma,\sigma} \neq 0$, then $\sigma(U) \neq \{0\}$ and $0 = \sigma(a^2y + ya^2) = [a^2,d(y)]_{\sigma,\sigma}$ for all $y \in U$. Thus, $\sigma(a^2) \in Z(R)$ by Lemma 4.1 and Lemma 2.8. So Substituting x by a in (4.12), we get $[ay+ya,d(a)]_{\sigma,\sigma} = 0$ for all $y \in U$. Putting ay instead of y, we obtain

$$\begin{aligned}
0 &= [a(ay+ya), d(a)]_{\sigma,\sigma} = \sigma(a)[ay+ya, d(a)]_{\sigma,\sigma} + [a,d(a)]_{\sigma,\sigma}\sigma(ay+ya) \\
&= [a,d(a)]_{\sigma,\sigma}\sigma(ay+ya).
\end{aligned}$$

Since, $[a,d(a)]_{\sigma,\sigma}$ is not a zero divisor, we have $\sigma(a)\sigma(y) - \sigma(y)\sigma(a) = 0$ for all $y \in U$. It follows that $\sigma(a)$ centralizes $\sigma(U) \neq \{0\}$. Lemma 4.1 and Lemma 2.8 implies $\sigma(a) \in Z(R)$. But that implies $[a,d(a)]_{\sigma,\sigma} = 0$, a contradiction. Therefore, $[x,d(x)]_{\sigma,\sigma} = 0$ for all $x \in U$ and R is commutative by Theorem 4.3.

The proof when U is a non-zero left ideal of R is similar. ∎

We finish this section by studying the commutativity of a prime ring R admitting a non-zero (σ,τ)-derivation d and satisfying the condition $d(x^2) \in Z(R)$ for all $x \in R$.

Proposition 4.2. *Let R be a prime ring with a non-zero (σ,τ)-derivation d such that τ is an automorphism and $d(x^2)=0$ for all $x\in R$. Then R is a commutative ring of characteristic 2.*

Proof. From $d((x+y)^2)=0$, we have $\sigma(x+y)d(x+y)=-d(x+y)\tau(x+y)$ for all $x,y\in R$. So $\sigma(x)d(x)+\sigma(x)d(y)+\sigma(y)d(x)+\sigma(y)d(y)=-d(x)\tau(x)-d(x)\tau(y)-d(y)\tau(x)-d(y)\tau(y)$. Using $\sigma(x)d(x)=-d(x)\tau(x)$ and $\sigma(y)d(y)=-d(y)\tau(y)$, we get $\sigma(x)d(y)+\sigma(y)d(x)=-d(x)\tau(y)-d(y)\tau(x)$ and then

$$d(xy)=-d(yx) \quad \text{for all} \quad x,y\in R.$$

Therefore, R is a commutative ring of characteristic 2 by Lemma 2.14. ∎

Theorem 4.6. *Let R be a prime ring with $2R\neq\{0\}$ and a non-zero (σ,τ)-derivation d such that σ and τ are automorphisms and $d(x^2)\in Z(R)$ for all $x\in R$. Then R is a commutative ring.*

Proof. From $d((x+y)^2)=\sigma(x+y)d(x+y)+d(x+y)\tau(x+y)\in Z(R)$ for all $x,y\in R$, we have $\sigma(x)d(x)+\sigma(x)d(y)+\sigma(y)d(x)+\sigma(y)d(y)+d(x)\tau(x)+d(x)\tau(y)+d(y)\tau(x)+d(y)\tau(y)\in Z(R)$. Using $\sigma(x)d(x)+d(x)\tau(x)\in Z(R)$, $\sigma(y)d(y)+d(y)\tau(y)\in Z(R)$ and that $Z(R)$ is a subring of R, we get $\sigma(x)d(y)+d(x)\tau(y)+\sigma(y)d(x)+d(y)\tau(x)\in Z(R)$ for all $x,y\in R$. It follows that $d(xy)+d(yx)\in Z(R)$ for all $x,y\in R$. If $Z(R)=\{0\}$, then R is a commutative ring of characteristic 2 by Lemma 2.14 and then $R=\{0\}$ and $d=0$, a contradiction. So there exists $c\in Z(R)-\{0\}$ such that $d(cy)+d(yc)=2d(cy)\in Z(R)$ for all $y\in Z(R)$. Thus,

$$d(cy)\in Z(R) \quad \text{for all} \quad y\in R \quad \text{and for all} \quad c\in Z(R)-\{0\} \tag{4.14}$$

by Proposition 4.1(ii). It follows that $d(ccc)=\sigma(c)d(cc)+d(c)\tau(cc)\in Z(R)$. Since $\sigma(c)d(cc)\in Z(R)$, we have $d(c)\tau(cc)\in Z(R)$ as $Z(R)$ is a subring of R. Using Lemma 2.3, Lemma 2.4 and τ is an automorphism, we get that $d(c)\in Z(R)$ for all $c\in Z(R)-\{0\}$.

If $d(Z(R))\neq\{0\}$, then there exists $c\in Z(R)-\{0\}$ such that $d(c)\in Z(R)-\{0\}$. From (4.14), we have $d(ccy)=\sigma(c)d(cy)+d(c)\tau(cy)\in Z(R)$. But $\sigma(c)d(cy)\in Z(R)$, so $d(c)\tau(cy)\in Z(R)$ for all $y\in R$. Using that $d(c),\tau(c)\in Z(R)-\{0\}$ and Lemma 2.3, we obtain $\tau(R)\subseteq Z(R)$. Therefore, R is a commutative ring since τ is onto.

If $d(Z(R))=\{0\}$, then for all $c\in Z(R)-\{0\}$, (4.14) implies

$$d(cy)=\sigma(c)d(y)+d(c)\tau(y)=\sigma(c)d(y)\in Z(R) \quad \text{for all} \quad y\in R.$$

Since σ is an automorphism, we have $d(R)\subseteq Z(R)$ by Lemma 2.3. Therefore, R is a commutative ring by Lemma 2.12. ∎

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