JOINT DISTRIBUTIONS

Outlines

-Discrete/Continuous Random Bivariate Variables

- Joint Probability Distributions

- Marginal Probability Distributions

- Conditional Probability Distributions

- Independence, Covariance and Correlation

-Random vector

Introduction
Bivariate Frequency Distributions

- For example, suppose you throw two coins, X and Y, simultaneously and record the outcome as an ordered pair of values. Imagine that you threw the coin 8 times, and observed the following (1=Head, 0 = Tail)

(X,Y)	***f***
(1,1)	2
(1,0)	2
(0,1)	2
(0,0)	2

- To graph the bivariate distribution, you need a 3 dimensional plot, although this can be drawn in perspective in 2 dimensions

Joint distribution of bivariate random variables

- In general, if X and Y are two random variables, the probability distribution that defines their simultaneous behavior is called a **joint probability distribution.**
- For example: X : the length of one dimension of an injection-molded part, and Y : the length of another dimension. We might be interested in

$$P(2.95 \le X \le 3.05 \text{ and } 7.60 \le Y \le 7.80).$$

Discrete case

Definition: If X and Y are discrete RV's, then (X, Y) is called a **jointly discrete bivariate RV**.

The joint (or bivariate) pmf is

$$f(x,y) = \Pr(X = x, Y = y).$$

Properties:

(1) $0 \le f(x,y) \le 1$.

(2) $\sum_x \sum_y f(x,y) = 1$.

(3) $A \subseteq \Re^2 \Rightarrow \Pr((X,Y) \in A) = \sum\sum_{(x,y)\in A} f(x,y)$.

Continuous case

Definition: If X and Y are cts RV's, then (X, Y) is a **jointly cts RV** if there exists a function $f(x, y)$ such that

Properties:

(1) $f(x, y) \geq 0$, $\forall x, y$.

(2) $\int\int_{\Re^2} f(x, y)\, dx\, dy = 1$.

(3) $\Pr(A) = \Pr((X, Y) \in A) = \boxed{\int\int_A f(x, y)\, dx\, dy.}$

is the volume between A and f(x,y)

In this case, $f(x, y)$ is called the **joint pdf**.

Marginal Probability Distribution

Discrete case

Definition: If X and Y are jointly discrete, then the **marginal pmf's** of X and Y are, respectively,

$$f_X(x) = \sum_y f(x,y)$$

and

$$f_Y(y) = \sum_x f(x,y)$$

Continuous case

Definition: If X and Y are jointly continuous random variable, then the **marginal pdf's** of X and Y are, respectively;

$$f_X(x) = \int_y f(x,y)dy$$

and

$$f_Y(y) = \int_x f(x,y)\, dx$$

General Example for discrete bivariate random variables

If X and Y are two discrete random variables ,their joint distribution may be represented by a formula or a table below:

	Y1	Y2	….	Yn	Marginal pdf of X g(X)
X1	f(x1,y1)	f(x1,y2)	….	f(x1,yn)	g(x1)
X2	f(x2,y1)	f(x2,y2	…	f(x2,yn)	g(x2)
…..	….	….	….	….	…
Xm	f(xm,y1)	f(xm,y2	….	f(x2,yn)	g(xm)
Marginal pdf of Y h(y)	h(y1)	h(y2)	…	h(yn)	Total =1

Discrete case example

	X=1	X=3	X=7	P(Y=y)
Y=10	2/7	1/7	0	9/21
Y=15	1/3	1/21	0	8/21
Y=20	1/7	0	1/21	4/21
P(X=x)	16/21	4/21	1/21	1

The marginal pmf's of X and Y respectively, are given by:

$$P(X=x)=f_X(x)=\begin{cases}\frac{16}{21}, & x=1\\ \frac{4}{21}, & x=3\\ \frac{1}{21}, & x=7\end{cases},\qquad P(Y=y)=f_Y(y)=\begin{cases}\frac{9}{21}, & y=10\\ \frac{8}{21}, & y=15\\ \frac{4}{21}, & y=20\end{cases}$$

where 16/21+4/21+1/21=1 **Also,** **9/21+8/21+4/21=1**

P(X=2)=? **P(Y=15)=?** **P(X >1, Y >10)=?**

Discrete case example (Read)

Example: 3 sox in a box (numbered 1,2,3). Draw 2 sox at random w/o replacement. $X =$ # of first sock, $Y =$ # of second sock. The joint pmf $f(x, y)$ is

	$X=1$	$X=2$	$X=3$	$\Pr(Y=y)$
$Y=1$	0	1/6	1/6	1/3
$Y=2$	1/6	0	1/6	1/3
$Y=3$	1/6	1/6	0	1/3
$\Pr(X=x)$	1/3	1/3	1/3	1

$\Pr(X=x)$ is the "marginal" distribution of X.

$\Pr(Y=y)$ is the "marginal" distribution of Y.

By the law of total probability,

$$\Pr(X=1) = \sum_{y=1}^{3} \Pr(X=1, Y=y) = 1/3.$$

P(X=2)=? P(Y=3)=? P(X>1, Y>2)=?

In addition,

$$
\begin{aligned}
&\Pr(X \geq 2, Y \geq 2) \\
&= \sum_{x \geq 2} \sum_{y \geq 2} f(x,y) \\
&= f(2,2) + f(2,3) + f(3,2) + f(3,3) \\
&= 0 + 1/6 + 1/6 + 0 \;=\; 1/3.
\end{aligned}
$$

Continuous case example: Read

Example: Suppose that

$$
f(x,y) = \begin{cases} 4xy & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$

Find the prob (volume) of the region $0 \leq y \leq 1 - x^2$.

$$
\begin{aligned}
V &= \int_0^1 \int_0^{1-x^2} 4xy\, dy\, dx \\
&= \int_0^1 \int_0^{\sqrt{1-y}} 4xy\, dx\, dy \\
&= 1/3.
\end{aligned}
$$

Example of joint density for continuous r.v.'s

- Let the joint density of X and Y be

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y}, & 0 < x < \infty, \quad 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

- Prove that

(1) $P\{X>1, Y<1\} = e^{-1}(1 - e^{-2})$

(2) $P\{X<Y\} = 1/3$

(3) $F_X(a) = 1 - e^{-a}$, $a > 0$, and 0 otherwise.

Note: Going from cdf's to pdf's (continuous case).

1-dimension: $f(x) = F'(x) = \frac{d}{dx} \int_{-\infty}^{x} f(t)\, dt.$

2-D: $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y) = \frac{\partial^2}{\partial x \partial y} \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\, dt\, ds.$

Bivariate Cumulative distribution Functions (CDF's)

Definition: The **joint (bivariate) cdf** of X and Y is

$F(x,y) \equiv P(X \leq x, Y \leq y)$, for all x, y.

$$F(x,y) = \begin{cases} \sum \sum_{s \leq x, t \leq y} f(s,t) & \text{discrete} \\ \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\, ds\, dt & \text{continuous} \end{cases}$$

Prooerties

- $F(x,y)$ is non-decreasing in both x and y.
- $\lim_{x \to -\infty} F(x,y) = \lim_{y \to -\infty} F(x,y) = 0$.
- $\lim_{x \to \infty} F(x,y) = F_Y(y) = \Pr(Y \leq y)$
- $\lim_{y \to \infty} F(x,y) = F_X(x) = \Pr(X \leq x)$.
- $F(x,y)$ is cts from the right in both x and y.

Exercise: Air Conditioner Maintenance (صيانة مكيف الهواء)

A company that services air conditioner units in residences and office blocks is interested in how to schedule its technicians in the most efficient manner

- The random variable X, taking the values 1,2,3 and 4, is the service time in hours
- The random variable Y, taking the values 1,2 and 3, is the number of air conditioner units

Y= number of units	X=service time			
	1	**2**	**3**	**4**
1	**0.12**	**0.08**	**0.07**	**0.05**
2	**0.08**	**0.15**	**0.21**	**0.13**
3	**0.01**	**0.01**	**0.02**	**0.07**

- Joint p.m.f and the total probability

$$\sum_i \sum_j p_{ij} = 0.12 + 0.08 + \cdots . + 0.07 = 1$$

- Joint cumulative distribution function

$$F(2,2) = p_{11} + p_{12} + p_{21} + p_{22}$$
$$= 0.12 + 0.18 + 0.08 + 0.15$$
$$= 0.43$$

Conditional Probability Distributions (1/3)

- **Conditional probability distributions**
  - The probability of the random variable X under the knowledge provided by the value of Y is given by
  - **Discrete case**

$$p_{i|j} = P(X = i \mid Y = j) = \frac{P(X = i, Y = j)}{P(Y = j)} = \frac{p_{ij}}{p_{+j}}$$

  - **Continuous case**

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

  - The conditional probability distribution is a **probability distribution.**

Because a conditional probability mass function $f_{Y|x}(y)$ is a probability mass function for all y in R_x, the following properties are satisfied:

(1) $f_{Y|x}(y) \geq 0$

(2) $\sum_{R_x} f_{Y|x}(y) = 1$

(3) $P(Y = y \mid X = x) = f_{Y|x}(y)$ (5-5)

Conditional Probability Distributions (2/3)

Conditional Mean and Variance

Definition

Let R_x denote the set of all points in the range of (X, Y) for which $X = x$. The **conditional mean** of Y given $X = x$, denoted as $E(Y|x)$ or $\mu_{Y|x}$, is

$$E(Y|x) = \sum_{R_x} y f_{Y|x}(y) \qquad (5\text{-}6)$$

and the **conditional variance** of Y given $X = x$, denoted as $V(Y|x)$ or $\sigma^2_{Y|x}$, is

$$V(Y|x) = \sum_{R_x} (y - \mu_{Y|x})^2 f_{Y|x}(y) = \sum_{R_x} y^2 f_{Y|x}(y) - \mu^2_{Y|x}$$

Example (Exercise: **Air Conditioner**)

- **Marginal probability distribution of Y**

$$P(Y=3) = p_{+3} = 0.01 + 0.01 + 0.02 + 0.07 = 0.11$$

- **Conditional distribution of X**

$$p_{1|Y=3} = P(X=1 | Y=3) = \frac{p_{13}}{p_{+3}} = \frac{0.01}{0.11} = 0.091$$

Conditional Probability Distributions (3/3)

- Example: The marginal probability distribution for X and Y.

y=number of times city name is stated	x=number of bars of signal strength			
	1	2	3	Marginal probability distribution of Y
4	0.15	0.1	0.05	0.3
3	0.02	0.1	0.05	0.17
2	0.02	0.03	0.2	0.25
1	0.01	0.02	0.25	0.28
	0.2	0.25	0.55	
	Marginal probability distribution of X			

$$P(Y=1 \mid X=3) = P(X=3, Y=1)/P(X=3)$$
$$= f_{x,y}(3,1)/f_x(3) = 0.25/0.55 = 0.454$$

$$E(Y \mid 1) = \sum_y y f_{Y|1}(y)$$
$$= 1(0.05) + 2(0.1) + 3(0.1) + 4(0.75) = 3.55$$

$$V(Y \mid 1) = \sum_y (y - \mu_{Y|x})^2 f_{Y|1}(y)$$
$$= (1-3.55)^2 0.05 + (2-3.55)^2 0.1 + (3-3.55)^2 0.1 + (4-3.55)^2 0.75$$
$$= 0.748$$

Expected Values for Jointly Distributed Continuous R.V.s

- Let X and Y be continuous random variables with joint probability density function f(x, y). We define E(X) and E(Y) as

$$\mathrm{E(X)} = \int_{-\infty}^{\infty} x \mathrm{f_X}(x)\mathrm{d}x \text{ and } \mathrm{E(Y)} = \int_{-\infty}^{\infty} y \mathrm{f_Y}(y)\mathrm{d}y.$$

- Example. For the random variables X and Y from the previous slide, $\mathrm{f_X}(x) = \mathrm{e}^{-x}, x > 0$ and $\mathrm{f_Y}(y) = 2\mathrm{e}^{-2y}, y > 0.$

  That is, X and Y are exponential random variables. It follows that

$$\mathrm{E(X)} = 1 \text{ and } \mathrm{E(Y)} = \frac{1}{2}.$$

Independence , Covariance and Correlation (1/13)

Independence

Discrete case

Two random variables X and Y are said to be independent if and only if

P[X=x,Y=y] = P[X=x] P[Y=y] *for all real numbers x and y.*

This definition of independence for discrete random variables translates into the statement that X and Y are independent if and only if a cell value is the product of the row total times the column total. i.e

$$p_{ij} = p_{i+} p_{+j} \quad \text{for all values } i \text{ of } X \text{ and } j \text{ of } Y$$

Continuous case

For continuous random variables, the condition for independence of X and Y becomes X and Y are independent if and only if

$$f(x, y) = f_X(x) f_Y(y) \quad \textit{for all } x \textit{ and } y$$

The marginal density functions can be multiplied together to produce the joint density function. Thus the random variables X and Y are independent.

Independence , Covariance and Correlation (2/13)

Example: (Discrete case)

Are the random variables X and Y described above with the following joint probability density table independent?

		Y Values				
		0	1	2	3	
X Values	0	1/8	0	0	0	1/8
	1	0	1/8	1/8	1/8	3/8
	2	0	1/4	1/8	0	3/8
	3	0	1/8	0	0	1/8
		1/8	1/2	1/4	1/8	

The random variables are not independent because, for example P[X=0,Y=1] = 0 but P[X=0] = 1/8 and P[Y=1] = 4/8.

Example: (continuous case)

For the joint density function f(x,y) = 1 for x on [0,1] and y on [0,1] and 0 otherwise, the marginal density function of X, $f_X(x)$ = 1 for x on [0,1] and the marginal density function of Y, $f_Y(y)$ = 1 for y on [0,1]. The marginal density functions can be multiplied together to produce the joint density function. Thus the random variables X and Y are independent

Functions of Independent Random Variables

- **Theorem**. Let X and Y be independent random variables and let g and h be real valued functions of a single real variable. Then

(i) g(X) and h(Y) are also independent random variables

(ii) $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.

- **Example**. If X and Y are independent, then $E[(\sin X)e^{Y}] = E[\sin X]E[e^{Y}]$.

Independence , Covariance and Correlation (4/13)

Covariance

$$\mathrm{Cov}(X,Y) = E((X-E(X))(Y-E(Y)))$$
$$= E(XY) - E(X)E(Y)$$

$$\begin{aligned}\mathrm{Cov}(X,Y) &= E((X-E(X))(Y-E(Y))) \\ &= E(XY - XE(Y) - E(X)Y + E(X)E(Y)) \\ &= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) \\ &= E(XY) - E(X)E(Y)\end{aligned}$$

- May take any positive or negative numbers.
- Independent random variables have a covariance of zero
- What if the covariance is zero?

$$E(XY)=E(X)E(Y)$$

Independence , Covariance and Correlation (5/13)

General Properties of (Cov(X, Y))

$$4-Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$$

$$5-Var\left(\sum_{i=1}^{n} X_i\right) = Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j}\sum Cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j}\sum Cov(X_i, X_j)$$

Independence , Covariance and Correlation (6/13)

- **Properties of** $Cov(X_1, X_2)$

$$Cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$$

, Cov(ax, bY) = ab Cov(X,Y)

$$\Rightarrow Cov(X_1, X_2) = Cov(X_2, X_1)$$

$$Cov(X_1, X_1) = Var(X_1) \quad Cov(X_2, X_2) = Var(X_2)$$

$$Cov(X_1 + X_2, X_1) = Cov(X_1, X_1) + Cov(X_2, X_1)$$

$$Cov(X_1 + X_2, X_1 + X_2) = Cov(X_1, X_1) + Cov(X_1, X_2) +$$

$$Cov(X_2, X_1) + Cov(X_2, X_2)$$

$$\Rightarrow Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$$

Independence , Covariance and Correlation (7/13)

General Properties of (Cov(X, Y))

$$4 - Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$$

$$5 - Var\left(\sum_{i=1}^{n} X_i\right) = Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j}\sum Cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j}\sum Cov(X_i, X_j)$$

Independence , Covariance and Correlation (8/13)

- **Example** (Air conditioner maintenance)

$$E(X) = 2.59, \qquad E(Y) = 1.79$$

$$E(XY) = \sum_{i=1}^{4} \sum_{j=1}^{3} ijp_{ij}$$

$$= (1\times1\times0.12) + (1\times2\times0.08)$$

$$+\cdots+(4\times3\times0.07) = 4.86$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= 4.86 - (2.59\times1.79) = 0.224$$

Y= number of units	X=service time			
	1	**2**	**3**	**4**
1	**0.12**	**0.08**	**0.07**	**0.05**
2	**0.08**	**0.15**	**0.21**	**0.13**
3	**0.01**	**0.01**	**0.02**	**0.07**

Exercise: Find Var(X), Var (Y)

Independence , Covariance and Correlation (9/13)

Correlation

- The correlation is a measure of the **linear** relationship between X and Y. It is obtained by dividing the covariance by the product of the two standard deviations, i.e.,

$$\operatorname{Corr}(X,Y)=\frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}}$$

- The correlation assumes values between -1 and 1.
- A value close to 1 implies a strong **positive** relationship.
- A value close to -1 implies a strong **negative** relationship.
- A value close to zero implies little or no relationship.
- The independent random variables have a correlation of zero.

Independence , Covariance and Correlation (10/13)

An important implication of independence

Suppose that the components X and Y of the discrete bivariate random variable (X,Y) are independent. Then its covariance is zero.

Always

$E(X-\mu)=0$

$$\sigma_{XY}=\mathrm{cov}(X,Y)$$

$$=\sum_{x\in X(\Omega)}\sum_{y\in Y(\Omega)}(x-\mu_X)(y-\mu_Y)P(X=x\wedge Y=y)$$

$$=\sum_{x\in X(\Omega)}\sum_{y\in Y(\Omega)}(x-\mu_X)(y-\mu_Y)P(X=x)P(Y=y)$$

$$=\sum_{x\in X(\Omega)}(x-\mu_X)P(X=x)\left[\sum_{y\in Y(\Omega)}(y-\mu_Y)P(Y=y)\right]$$

$$=E(X-\mu_X)E(Y-\mu_Y)$$

$$=0$$

Independence , Covariance and Correlation (11/13)

The continuous case

$$\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - \mu_X)(y - \mu_Y)f_{XY}(x, y)dydx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - \mu_X)(y - \mu_Y)f(x)f(y)dydx$$

$$= \int_{-\infty}^{\infty}(x - \mu_X)f(x)\int_{-\infty}^{\infty}(y - \mu_Y)f(y)dydx$$

$$= \int_{-\infty}^{\infty}(x - \mu_X)f(x)\left[\int_{-\infty}^{\infty}(y - \mu_Y)f(y)dy\right]dx$$

$$= \left[\int_{-\infty}^{\infty}(y - \mu_Y)f(y)dy\right]\int_{-\infty}^{\infty}(x - \mu_X)f(x)dx$$

$$= E(Y-\mu_Y)E(X-\mu_X)$$

$$= 0.$$

Independence , Covariance and Correlation (12/13)

- Example : (Air conditioner maintenance)

$$\mathrm{Var}(X) = 1.162, \quad \mathrm{Var}(Y) = 0.384$$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

$$= \frac{0.224}{\sqrt{1.162 \times 0.384}} = 0.34$$

Independence , Covariance and Correlation (13/13)

Moments of bivariate random variables

Joint moments of (X,Y):

$$EX,\ EY$$
$$EX^2,\ EXY,\ EY^2$$
$$EX^3,\ EX^2Y,\ EXY^2,\ EY^3$$
$$\vdots$$

Joint central moments of (X,Y):

$$E(X-EX),\ E(Y-EY)$$
$$E(X-EX)^2,\ E(X-EX)(Y-EY),\ E(Y-EY)^2$$
$$E(X-EX)^3,\ E(X-EX)^2(Y-EY),\ E(X-EX)(Y-EY)^2,\ E(Y-EY)^3$$
$$\vdots$$

- I read from my textbook that $\mathrm{cov}(X,Y)=0$ does not guarantee X and Y are independent. But if they are independent, their covariance must be 0. I could not think of any proper example yet; could someone provide one?

- Easy example: Let X be a random variable that is -1 or $+1$ with probability 0.5. Then let Y be a random variable such that $Y=0$ if $X=-1$, and Y is randomly -1 or $+1$ with probability 0.5 if $X=1$.
- Clearly X and Y are highly dependent (since knowing Y allows me to perfectly know X), but their covariance is zero: They both have zero mean, and
- $$\begin{aligned}\mathbb{E}[XY] &= (-1)\cdot 0 \cdot P(X=-1)\\ &+1 \cdot 1 \cdot P(X=1,Y=1)\\ &+1 \cdot(-1)\cdot P(X=1,Y=-1)\\ &=0.\end{aligned}$$

- Or more generally, take any distribution $P(X)$ and any $P(Y|X)$ such that $P(Y=a|X)=P(Y=-a|X)$ for all X (i.e., a joint distribution that is symmetric around the x axis), and you will always have zero covariance. But you will have non-independence whenever $P(Y|X)\neq P(Y)$; i.e., the conditionals are not all equal to the marginal. Or ditto for symmetry around the y axis.

Exercices

(1) Let the joint distribution of X and Y is given by :

$$f(x,y) = cxy,\ x=1,2,3;\ y=1,2,3.$$

Find

(i) the constant c,

(ii) (ii) $p(x=2,y=3)$,

(iii) $P(1 \le X \le 2, Y \le 2)$ $P(Y<2)$, $p(X=1)$ $p(Y=3)$

Find the marginal probability mass functions of X and Y above and determine whether x and Y are independent

(2) Let X, Y have joint density functions

$$f(x,y) = \begin{cases} c(x^2+y^2) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{, otherwise} \end{cases}$$

Determine

(a) the constant c

(b) $P(X< \frac{1}{2}, Y> \frac{1}{2})$, $P(\frac{1}{4} < X < \frac{3}{4})$

(d) $P(Y< \frac{1}{4})$

(e) Find the marginal distribution functions of X and Y.

) Are X, Y independent

(3) The joint distributions are given in each question. Find the conditional distribution of

(a) X given Y

(b) Y given X

2.58) $f(x,y) = \left(\frac{xy}{36}\right),$

$x = 1,\ 2,\ 3\ \text{and}\quad y = 1,\ 2,\ 3$

(2.59) $$f(x, y) = \begin{cases} \frac{3}{2}(x^2 + y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ o & \text{, otherwise} \end{cases}$$

(2.60) $$f(x, y) = \begin{cases} (x + y), & 0 \le x \le 1,\ 0 \le y \le 1 \\ o & \text{, otherwise} \end{cases}$$

(2.61) $$f(x, y) = \begin{cases} e^{-(x+y)}, & x \ge 0,\ y \ge 0 \\ o & \text{, otherwise} \end{cases}$$