Chapter 4(2)

Hypergeometric Distribution

Poisson Distribution

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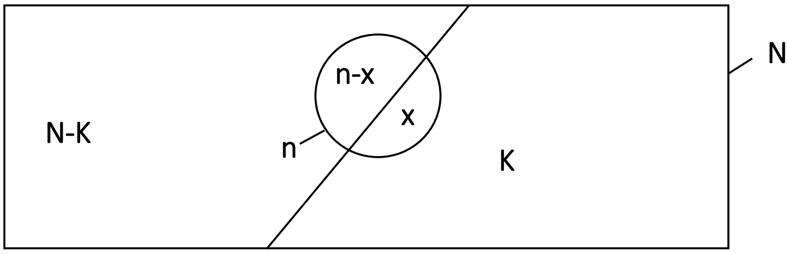

In a group of N objects, K are of Type I and $N-K$ are of Type II. If n objects are randomly chosen without replacement from the group of N, let X denote the number that are of Type I in the group of n. Thus, X has a hypergeometric distribution $X \sim H(N, n, K)$. The pmf for X is

$$f(x) = f(x; N, n, K) = \begin{cases} \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}; & x = Max[0, n-(N-K)], \dots, Min[n, K] \\ 0; & otherwise \end{cases}$$

Parameters of the Distribution: $N \in N^+$(population size), $n \in N^+$ (sample size), $K \in N^+$(population elements with a certain characteristic).

Characteristics of Hypergeometric Distribution

1. 'n' trials in a sample taken from a finite population of size N.
2. The population (outcome of trials) has two outcomes Success (S) and Failure(F).
3. Sample taken **without replacement**.
4. Trials are dependent.
5. The probability of success changes from trial to trial.

Mean and Variance

If X is a discrete random variable has hypergeometric distribution with parameters M, n, K then,

$$E(X) = \mu = n\frac{K}{N} \text{ and } V(x) = \sigma^2 = n\frac{k}{N}(1-\frac{k}{N})(\frac{N-n}{N-1})$$

EX

A wallet contains 3 $100 bills and 5 $1 bills. You randomly choose 4 bills. What is the probability that you will choose exactly 2 $100 bills?

Solution :

(a) $\boldsymbol{N = 8 , n = 4 , K = 3 , N - K = 5}$

$$\boldsymbol{P(X=2) = \frac{\binom{3}{2}\binom{5}{2}}{\binom{8}{4}} = 0.4286}$$

(b) Find mean and variance

$$\boldsymbol{mean = \mu = n\frac{K}{N} = 4\,x\frac{3}{8} = 1.5}$$

$$\boldsymbol{Variance} = \sigma^2 = n\frac{k}{N}\left(1-\frac{k}{N}\right)\left(\frac{N-n}{N-1}\right) = 4x\frac{3}{8}x\frac{5}{8}x\frac{4}{7} = 0.5357$$

$$\boldsymbol{Standard\ deviation} = \sigma = \sqrt{\boldsymbol{Variance}} = \sqrt{0.5357} = 0.7319$$

Example 3.9

A box contains 6 blue and 4 red balls. An experiment is performed a ball is chosen and its color observed. Find the probability, that after 5 trials, 3 blue balls will have been chosen when

I. The balls are replaced (with replacement)

II. The balls not replaced (without replacement)

Solution

I. Let X represents the no. of blue balls in the sample. $\boldsymbol{X \sim Bin(5, 0.6)}$. So, we want to find

$$P(X=3) = \binom{5}{3}(0.6)^3(0.4)^2 = 0.3456.$$

II. Let Y represents the no. of blue balls in the sample. $\boldsymbol{Y \sim H(10, 5, 6)}$. So, we want to find

$$P(Y=3) = \frac{\binom{6}{3}\binom{4}{2}}{\binom{10}{5}} = 0.4762.$$

Poisson Distribution

The Poisson distribution is often used as a model for counting the number of events of a certain type that occur in a certain period of time (or space). If the r.v. X has Poisson distribution $X \sim Poisson(\lambda)$ then its pmf is given by

$$f(x) = f(x;\lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!}; & x = 0,1,2,\ldots \\ 0; & otherwise \end{cases}$$

Parameter of the Distribution: $\lambda > 0$ (The average)

Mean and Variance

If X is a discrete random variable has Poisson distribution with parameter λ then,

$$E(X) = V(x) = \lambda t.$$

For example

- The number of births per hour during a given day.
- The number of failures of a machine in one month.
- The number of typing errors on a page.
- The number of postponed baseball games due to rain.

Example

Suppose that X represents the number of customers arriving for service at bank in a one hour period, and that a model for X is the Poisson distribution with parameter λ. In general, for any time interval of length t, the number of customers arriving in that time interval has a Poisson distribution with parameter $\mu = \lambda t\ , t\ is\ time$

(a) X, the number of bank customers arriving in one hour, Suppose that $\lambda = 40$, It 'means that X has mean of 40'.

$$(here\ t = 1, \mu = \lambda t = 40x1 = 40)$$

(b) Y represents the number of customers arriving in 2 hours, then Y has a Poisson distribution with a parameter μ = 80.

$$(here\ t = 1, \mu = \lambda t = 40x2 = 80)$$

(c)Z represents the number of customers arriving during a 15-minute period ,then Z a Poisson distribution with parameter $40 \cdot \frac{1}{4} = 10$.

$$(here\ t = 1/4, \mu = \lambda t = 40x1/4 = 10)$$

So, In general, If W represents the number of customers arriving in t hours $W \sim Poisson(\lambda t)$ therefore,

$$f(w) = \frac{e^{-\lambda t}(\lambda t)^w}{w!}; \quad w = 0, 1, 2, \ldots.$$

<u>Example 3.11</u>

Suppose that the number of typing errors per page has a Poisson distribution with average 6 typing errors. What is the probability that

I. The number of typing errors in a page will be 7.
II. The number of typing errors in a page will be at least 2.
III. The number of typing errors in 2 pages there will be 10 typing errors.
IV. The number of typing errors in a half page there will be no typing errors.
V. Mean of typing errors in per 3 pages
VI. Standard deviation of typing errors in per 1/2 pages

<u>Solution</u>

I. Let X represents the no. of typing errors per page.

Therefore, $\lambda_X = 6 \Rightarrow X \sim Poisson(6)$.

$$P(X = 7) = \frac{e^{-6}6^7}{7!} = 0.1377.$$

II. $\boldsymbol{P(X \geq 2) = f(2) + f(3) + \cdots = 1 - P(X < 2) = 1 - [f(0) - f(1)]}$

$$= 1 - [\frac{e^{-6}6^0}{0!} - \frac{e^{-6}6}{1!}] = 0.9826.$$

III. Let Y represents the no. of typing errors in 2 pages.

Therefore, $\lambda_Y = \lambda_X t = 6 \cdot 2 = 12 \Rightarrow Y \sim Poisson(12)$.

$$P(Y = 10) = \frac{e^{-12}(12)^{10}}{10!} = 0.1048.$$

IV. Let Z represents the no. of typing errors in a half pages.

Therefore, $\lambda_Z = \lambda_X t = 6 \cdot \frac{1}{2} = 3 \Rightarrow Z \sim Poisson(3)$.

$$P(Z = 0) = \frac{e^{-3}3^0}{0!} = 0.0498.$$