Problem Set (2)

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PHYS 453: Quantum mechanics

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Problem 2.1. Given the Hamiltonian operator for a 2-level system :

$$\hat{H} = \begin{pmatrix} 2-\Omega & -i \\ i & 3 \end{pmatrix}. \tag{1}$$

Find the energy-levels of this system.

Problem 2.2. We have the eingenfunction $\psi(x) = Ax^{1/2}e^{-x/2\lambda}$ defined over the interval $[0, \infty]$, with eigenenergy $\epsilon = 0$.

(i) Find the value of A

(ii) Show that $\langle \hat{p} \rangle = 0$

(iii) Find the potential energy operator for this system

Problem 2.3. The Hamiltonian operator for an electron in a magnetic field $\vec{B} = B_z \vec{e}_z$ is given by

$$\hat{H} = -\mu_B \frac{\sigma_z B_z}{2}. \tag{2}$$

Where σ_z is Pauli matrix and μ_B is a positive constant.

(i) Calculate $\langle \hat{H} \rangle$ with respect to the state $|\chi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(ii) Show that if we transformed the Hamiltonian as the following:

$$\hat{H} \to (I - \sigma_y^\dagger)\hat{H}(I + \sigma_y), \tag{3}$$

It remains unchanged.
(This corresponds to rotating the system by the y-axis)

Problem 2.4. Prove the first part of Ehrenfest theorem,

$$m\frac{d}{dt}\left(\langle x \rangle\right) = \langle p \rangle. \tag{4}$$

Using Shrödinger's equation

Useful formulae

† **Pauli Matrices**:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

† **Gamma function**

$$\Gamma(n) = (n-1)! \quad \text{if n is integer}$$
$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx \quad z \in \mathbb{C}$$

Particular values

$$\Gamma\left(\tfrac{n}{2}\right) = \sqrt{\pi}\frac{(n-2)!!}{2^{\frac{n-1}{2}}}$$

† **Trigonometric identities**

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$
$$cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

† **Integrals involving exponential functions**

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}\, dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \quad (a > 0)$$

$$\int_0^\infty x^n e^{-ax^2}\, dx = \begin{cases} \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{2a^{\frac{n+1}{2}}} & (n > -1,\ a > 0) \\ \dfrac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\dfrac{\pi}{a}} & (n = 2k,\ k \text{ integer},\ a > 0) \\ \dfrac{k!}{2a^{k+1}} & (n = 2k+1,\ k \text{ integer},\ a > 0) \end{cases}$$