Exercises

Exercise 1: If $X=\mathbb{R}$. Show that the Borel σ-algebra is generated by each of the following collection of sets:

(1) $C_1 = \{ (a,b) \ ; a,b \in \mathbb{R} \}$

(2) $C_2 = \{ [a,b] \ ; a,b \in \mathbb{R} \}$

(3) $C_3 = \{ (a,b] \ ; a,b \in \mathbb{R} \}$

(4) $C_4 = \{ (a,\infty) \ ; a \in \mathbb{R} \}$.

$\mathcal{B}(\mathbb{R}) = \sigma(C_1) = \sigma(C_2) = \sigma(C_3) = \sigma(C_4)$

Exercise 2: For all $n \in \mathbb{N}-\{0\}$, define $C_{2n} = [-2, 1+\frac{1}{2n})$ and $C_{2n+1} = [-3-\frac{1}{2n+1}, 0]$. Show that

$$\overline{C_n} := \limsup C_n = [-3,1] \text{ and } \underline{C_n} := \liminf_n C_n = [-2,0].$$

Exercise 3: Let $(A_n)_{n\geq 1}$ be a sequence of measurable sets such that $\sum_n \mu(A_n) < \infty$

Show that the set of points which belong to infinite number of A_n's has measure zero.

Hint: prove that the set considered is $\bigcap_{n=1}^{\infty} \left(\bigcup_{k\geq n} A_k \right)$.

Exercise 4:

Let E be infinite uncountable set. For all subset A of E put $\mu(A)=0$ if A is at most countable and $\mu(A)=\infty$ otherwise. Does the mapping μ define a measure on $\mathcal{P}(E)$?

Answer

Exercise 1 :

(1) Let $\mathcal{G}$ be the collection of open sets. Then $\mathcal{C}_1 \subset \mathcal{G} \subset \sigma(\mathcal{G})$
$\sigma(\mathcal{G})$ is the Borel σ-algebra and contains $\mathcal{C}_1$.
Since $\sigma(\mathcal{C}_1)$ is the intersection of all all σ-algebras containing $\mathcal{C}_1$ then $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{G}) = \mathcal{B}(\mathbb{R})$
To get the reverse inclusion, if G is open, it is the countable union of open intervals. so $G \subset \sigma(\mathcal{C}_1)$. But $\sigma(\mathcal{G})$ is the intersection of all σ-algebras containing $\mathcal{G}$. Then $\sigma(\mathcal{C}_1)$ is one such so $\sigma(\mathcal{G}) \subset \sigma(\mathcal{C}_1)$.

(2) If $[a,b] \in \mathcal{C}_2$ then $[a,b] = \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n}) \in \sigma(\mathcal{G})$
So $\mathcal{C}_2 \subset \sigma(\mathcal{G})$ and by an argument similar to that (1), we conclude $\sigma(\mathcal{C}_2) \subset \sigma(\mathcal{G})$.
If $(a,b) \in \mathcal{C}_1$, choose $n_0 \geq \frac{2}{b-a}$ and note $(a,b) = \bigcup_{n=n_0}^{\infty} [a+\frac{1}{n}, b-\frac{1}{n}] \in \sigma(\mathcal{C}_2)$

So the Borel σ-algebra which is equal to $\sigma(\mathcal{C}_1)$ by part (1) is contained in $\sigma(\mathcal{C}_2)$.

(3) The proof here is similar to (2), using $(a,b] = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n})$ and $(a,b) = \bigcup_{n=n_0}^{\infty} (a, b-\frac{1}{n}]$ provided n_0 is taken large enough.

(4) The proof of this comes from (3), using that : $(a,b] = (a,\infty) - (b,\infty)$ and $(a,\infty) = \bigcup_{n=1}^{\infty} (a, a+n)$.

Exercise 2 :

- $\underline{C_n} := \liminf_n C_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{k \geq n} C_k \right)$

Let $n \geq 1$,
$$\bigcap_{k \geq n} C_k = \left(\bigcap_{\substack{k \geq n \\ k \text{ even}}} C_k \right) \cap \left(\bigcap_{\substack{k \geq n \\ k \text{ odd}}} C_k \right) = [-2,1] \cap [-3,0].$$
$$= [-2,0]$$
So $\underline{C_n} := \liminf_n C_n = [-2,0]$.

- $\overline{C_n} := \limsup_n C_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} C_k \right)$

Let $n \geq 1$,
$$\bigcup_{k \geq n} C_k = \left(\bigcup_{\substack{k \geq n \\ k \text{ even}}} C_k \right) \cup \left(\bigcup_{\substack{k \geq n \\ k \text{ odd}}} C_k \right) = [-2, 1+\frac{1}{2n}) \cup [-3-\frac{1}{2n+1}, 0].$$

For $n \geq 1$; $\bigcup_{k \geq n} C_k = \left[-3 - \frac{1}{2n+1} ; 1 + \frac{1}{2n}\right)$

$$\bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} C_k \right) = \bigcap_{n=1}^{\infty} \left[-3 - \frac{1}{2n+1} ; 1 + \frac{1}{2n}\right) = [-3, 1]$$

So $\overline{C_n} := \limsup_n C_n = [-3, 1]$.

<u>Exercise 3</u> :

It is clear that if $x \in \bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} A_k \right)$ then x belongs to infinitely many A_n's.

Now let x belongs to infinitely many A_n's. Then for any n there is a $k \geq n$ such that $x \in A_k$. So we have proved that the set considered is $A := \bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} A_k \right)$.

For every $n \geq 1$, one has $\mu\left(\bigcup_{k \geq n} A_k \right) \leq \sum_{k \geq n} \mu(A_k)$ and consequently

$$\mu(A) = \lim_{n \to \infty} \mu\left(\bigcup_{k \geq n} A_k \right) \leq \lim_{n \to \infty} \underbrace{\left(\sum_{k \geq n} \mu(A_k) \right)}_{\text{the remainder of a convergent series.}} = 0$$

We deduce $\mu(A) = 0$.