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College of sciences
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List of exercises $n°6$ (Math 580 Theory Measure I)

Exercise 1:

Let $(X, \mathcal{A}, \mu)$ be a probability space and f, g be two Borelian, positives functions such that $f.g \geq 1$. Show that

$$\left(\int_X f(x)d\mu(x)\right) . \left(\int_X g(x)d\mu(x)\right) \geq 1.$$

Exercise 2:

Let λ be the Lebesgue measure on $X = [0,1]$ and $(f_n)_{n\geq 1}$ be a sequence of measurable functions on X with real values and satisfying $\lim_{n\to\ +\infty} \int_X |f_n(x)|^3 d\lambda(x) = 0$. Prove that

$$\lim_{n\to+\infty} \int_X \frac{f_n(x)}{\sqrt{x}} d\lambda(x) = 0.$$

Exercise 3:

1. Let $(f_n)_n$ be a sequence of functions from $L^p(X, \mu)$, $p \geq 1$ such that
   (i) (f_n) converges to f almost everywhere.
   (ii) $\lim_{n\to+\infty} ||f_n||_p = ||f||_p$.
   We define the sequence $(\phi_n)_n$ by:
   $$\phi_n(x) = 2^{p-1}\left(|f(x)|^p + |f_n(x)|^p\right) - |f(x) - f_n(x)|^p.$$
   (a) Prove that $\phi_n \geq 0$ for all n.
   (b) Use the Fatou's lemma to prove that $\lim_{n\to+\infty} f_n = f$ in L^p.

2. Give a sequence $(f_n)_n$ of functions in $L^1(\mathbb{R})$ which converges to 0 almost everywhere but $(f_n)_n$ does not convergent in L^1.

Exercise 4:

Let $(X, \mathcal{A}, \mu)$ be a measure space, f be a function in $L^1(X, \mathcal{A}, \mu)$ and $(f_n)_{n\geq 1}$ be a sequence of functions in $L^1(X, \mathcal{A}, \mu)$ such that $\lim_{n\to+\infty} \int_X f_n(x)d\mu(x) = \int_X f(x)d\mu(x)$.

1. Show that if for all $n \geq 1$, the function f_n is positive and if the sequence (f_n) converges a.e to f then (f_n) converges to f in L^1.
   Hint: Consider $g_n := \min(f, f_n)$.
   Now we consider the Lebesgue space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and the sequence (f_n) defined by:
   $$f_n = n\chi_{(0,\frac{1}{n})} - n\chi_{(-\frac{1}{n},0)}.$$
2. Prove that (f_n) converges to 0 and $\lim\limits_{n\to+\infty} \int_{\mathbb{R}} f_n(x)d\lambda(x) = 0$.
3. Does the sequence (f_n) converges to 0 in L^p for $p \in [1, +\infty)$?

Exercise 5:
Let $(X, \mathcal{A}, \mu)$ be a probability space and f be a Borelian, positive, integrable function.

1. Use Hölder's inequality to prove that:
   if $\mu(\{f > 0\}) < 1$ then $\lim\limits_{p\to 0^+} ||f||_p = 0$.
2. Show that $\lim\limits_{p\to 0^+} \int_X f^p d\mu = \mu(\{f > 0\})$.
3. Show that for all $p \in (0, 1)$ and $\forall x \in (0, +\infty)$,
   $$\frac{|x^p - 1|}{p} \leq x + |\ln x|.$$
   We assume that $f > 0$ and $\ln f$ is also $\mu-$integrable.
4. Show that $\lim\limits_{p\to 0^+} \int_X \frac{f^p - 1}{p} d\mu = \int_X \ln(f)d\mu$.
5. Show that $\lim\limits_{p\to 0^+} ||f||_p = \exp\left(\int_X \ln(f)d\mu\right)$.

Exercise 6:
Let $p > 1$. For every function $f \in L^p(\mathbb{R}_+)$, we associate the function F defined on $(0, +\infty)$ by
$$F(x) = \frac{1}{x}\int_0^x f(t)dt.$$

1. Show that F is well-defined.

2. We suppose that $f \in \mathcal{C}_K(\mathbb{R}_+^*, \mathbb{R}_+)$[1]. Show that:

$$\int_0^{+\infty} (F(x))^p dx = -p \int_0^{+\infty} x(F(x))^{p-1} F'(x) dx \text{ and}$$

$$\int_0^{+\infty} (F(x))^p dx = \frac{p}{p-1} \int_0^{+\infty} f(x)(F(x))^{p-1} dx.$$

3. Deduce the Hardy's inequality:

$$||F||_p \leq \frac{p}{p-1} ||f||_p.$$

4. Prove the Hardy's inequality for the functions in $L^p(\mathbb{R}_+)$.

5. Show that the Hardy's inequality becomes equality if and only if $f \equiv 0$ almost everywhere.

6. Show that the constant $\frac{p}{p-1}$ can not replaced by another smallest constant.
   <u>*Hint:*</u> *Consider* $f(x) = \chi_{[1,A]}(x).x^{-1/p}$.

[1] $\mathcal{C}_K(\mathbb{R}_+^*, \mathbb{R}_+)$ is the set of all continuous, positive, functions with compact support $K \subset \mathbb{R}_+^*$.