King Saud University
College of sciences
Department of mathematics
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Dr. Borhen Halouani

List of exercises $n°5$ (Math 580 Theory Measure I)

Exercise 1:

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a monotone function.

1. Show that f is Riemann-integrable.
2. Show that f has at each point of $[a, b[$ (resp.$]a, b]$) a right limit (resp. a left limit).

Exercise 2:

Let $f,\ g : [a, b] \longrightarrow \mathbb{R}$ be two Riemann-integrable functions.

1. Show that $|f|$ is integrable and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.
2. Show that $\forall p \in \mathbb{N}$, $|f|^p$ is Riemann-integrable.
3. Show that $(f.g)$ is Riemann-integrable.

Exercise 3:

Let $(f_n)_n$ be a sequence of functions defined on $[0, 1]$ by $f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k}$.

Show that $(f_n)_n$ is a Cauchy sequence but it does not convergent in $(\mathcal{C}([0, 1] \to \mathbb{R}), N_1)$ where $\mathcal{C}\,([0, 1] \to \mathbb{R})$ is the set of all continuous functions on $[0, 1]$ and $N_1(f) = \int_0^1 |f(x)|dx$.

Exercise 4:

Give an example of measure μ on $\left(\mathbb{R}^2, \mathcal{B}^{\otimes^2}\right)$ which is not the tensorial product of two measures on $(\mathbb{R}, \mathcal{B})$.

Exercise 5:

1. Show that $\mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N}) = \mathcal{P}(\mathbb{N}^2)$.
2. Let μ be the counting measure of $\mathbb{N}$. Show that $\mu \otimes \mu$ is the counting measure of $\mathbb{N}^2$.

Exercise 6:
Let λ be the Lebesgue measure on $([0,1],\mathcal{B}([0,1]))$ and μ be the counting measure on $([0,1],\mathcal{P}([0,1]))$. Denote $\Delta=\{(x,x);x\in[0,1]\}$ be the diagonal of $[0,1]^2$.

1. Show that $\Delta\in\mathcal{B}\otimes\mathcal{P}$.
2. Compute $\int\left(\int\chi_\Delta(x,y)\ d\lambda(x)\right)d\mu(y)$ and $\int\left(\int\chi_\Delta(x,y)\ d\mu(y)\right)d\lambda(x)$.
3. Explain.

Exercise 7:
Let f_1 and f_2 be two functions defined on $\mathbb{R}^2$ by :

$$f_1(x,y)=\begin{cases}\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}\ ;\ f_2(x,y)=\begin{cases}\frac{x-y}{(x^2+y^2)^{3/2}} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

1. For $j=1,2$, compute: $\int_0^1\left(\int_0^1 f_j(x,y)dy\right)dx$ and $\int_0^1\left(\int_0^1 f_j(x,y)dx\right)dy$.
2. Conclude.

Exercise 8:
Let $f:\mathbb{R}^d\longrightarrow\mathbb{R}$ be a Borelian function and $\Gamma=\{(x,f(x));x\in\mathbb{R}^d\}$ its graph.

1. Show that Γ is a Borelian set of $\mathbb{R}^{d+1}$.
2. Show that Γ is a null set for the Lebesgue measure of $\mathbb{R}^{d+1}$.

Exercise 9:

1. Study the integrability of $f(x,y)=\frac{1}{(1+x+y)^\alpha}$ on $[0,\infty)^2$ where $\alpha\in\mathbb{R}$ is a parameter and compute its integral, if it exists.
2. Use the fact $\frac{1}{x}=\int_0^\infty e^{-xt}dt$ to prove that : $\lim_{a\to\infty}\int_0^a\frac{\sin x}{x}dx=\frac{\pi}{2}$.
3. Show that the function $\frac{\sin x}{x}$ is not integrable on $(0,\infty)$.
   (Hint: By contradiction, deduce that $\frac{\sin^2 x}{x}$ is integrable and prove that it follows a contradiction).
4. Let $0<a<b$ be 2 real numbers and f be the real function defined on $[0,1]\times[a,b]$ by $f(x,y)=x^y$. Show that f is Lebesgue integrable for the Lebesgue measure on $([0,1]\times[a,b],\mathcal{B}([0,1]\times[a,b]))$ an deduce $\int_0^1\frac{x^b-x^a}{\ln x}dx$.