King Saud University
College of sciences
Department of mathematics
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List of exercises $n^\circ 4$ (Math 580 Theory Measure I)

<u>Exercise 1:</u>

Let λ be the Lebesgue measure on the interval $[-1, 1]$. Let g be a function defined by : $g(x) = 1$ if $x \in [0, 1]$ and $g(x) = 0$ otherwise. Let $(f_n)_n$ be a sequence of functions defined as $f_n(x) = g(x)$ if n is even and $f_n(x) = g(-x)$ if n is odd. Prove that

$$\int \left(\liminf_{n\to\infty} f_n(x) \right) \, d\lambda < \liminf_{n\to\infty} \left(\int f_n(x) \, d\lambda \right).$$

<u>Exercise 2:</u>

Let $f : (E, \mathcal{E}) \longrightarrow (F, \mathcal{F})$ be a measurable map and μ be a bounded measure on $(E, \mathcal{E})$. Denote $f_*\mu$ the image measure of μ by f. It is the measure on $(F, \mathcal{F})$ defined by

$$f_*\mu(B) = \mu(f^{-1}(B)), \quad \forall B \in \mathcal{F}.$$

Show that the $\mathcal{F}-$measurable function $\phi : F \longrightarrow \mathbb{R}$ is $f_*\mu-$integrable if and only if $\phi \circ f$ is μ-integrable and deduce that

$$\int_E \phi \circ f \, d\mu = \int_F \phi \, d(f_*\mu).$$

<u>Exercise 3:</u>

Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a measurable function. Determine the limit of

$$\int_0^1 \frac{dt}{\sqrt{(f(t))^2 + \frac{1}{n}}}.$$

<u>Exercise 4:</u>

Determine the limit of the following integrals when n tends to $+\infty$:

$$u_n = \int_0^1 \frac{1 + nx^3}{(1 + x^2)^n} \, dx \; ; \; v_n = \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} \, dx \text{ and } w_n = \int_0^\infty \frac{\sin(\pi x)}{1 + x^n} \, dx.$$

Exercise 5:
Let a, b be two real numbers such $a < b$, and $f : (a, b) \longrightarrow \mathbb{R}$ be a bounded, integrable, Borelian function such that $\lim_{x \to a^+} f(x) = \gamma \in \mathbb{R}$.
Prove that for all $t \in (a, b)$, the function $x \longmapsto \frac{f(x)}{\sqrt{(x-a)(t-x)}}$ is integrable on (a, t) and compute

$$\lim_{t \to a^+} \int_a^t \frac{f(x)}{\sqrt{(x-a)(t-x)}}\, dx.$$

Exercise 6:

1. Let a and b be two strictly positif real numbers. For $x > 0$, let $f(x) = \frac{xe^{-ax}}{1-e^{-bx}}$. Show that

$$\int_0^{+\infty} f(x)dx = \sum_{n=0}^{\infty} \frac{1}{(a+nb)^2}.$$

2. Let μ be a measure on $\mathbb{R}$ such that the function $x \longmapsto e^{x^2}$ be $\mu-$integrable. Give an example of a measure for μ, and prove that for all complex number $z \in \mathbb{C}$, we have:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathbb{R}} x^n d\mu(x) = \int_{\mathbb{R}} e^{zx} d\mu(x).$$

3. Prove that the function $f : [1, \infty) \longrightarrow \mathbb{R}$ defined by $f(x) = \sum_{n=1}^{\infty} ne^{-nx}$ is integrable on $[1, \infty)$ for the Lebesgue measure and compute its integral.

Exercise 7:
Let f be the function defined by $f(x, t) = e^{-xt} \frac{\sin x}{x} \chi_{(0,\infty)}(x)$.

1. Prove that for every $t > 0$, the function $x \longmapsto f(x, t)$ is integrable for the Lebesgue measure on $\mathbb{R}$.
2. Prove that the function $F(t) = \int_{\mathbb{R}} f(x, t)\, dx$ is differentiable on $(0, \infty)$.
3. Determine the expression of F.
4. Can we deduce that the function $x \longmapsto \frac{\sin x}{x}$ is integrable on $(0, \infty)$?

Exercise 8:
Let $(E, \mathcal{E}, \mu)$ be a measure space and $f : E \longrightarrow \mathbb{C}$ be $\mu-$integrable function.

1. Let $B_n = \{x \in E; n-1 \leq |f(x)| \leq n\}$. Prove that $\mu(B_n) < \infty, \forall n \geq 2$.

2. Prove that $\sum_{n=2}^{\infty} n\mu(B_n) < \infty$.

3. Writting, for $N \geq 2$, $\sum_{n=2}^{N}\sum_{m=2}^{n}\frac{m^2}{n^2}\mu(B_m) = \sum_{m=2}^{N}\sum_{n=m}^{N}\frac{m^2}{n^2}\mu(B_m)$. Show that
$$\sum_{n=2}^{\infty}\sum_{m=2}^{n}\frac{m^2}{n^2}\mu(B_m) < \infty.$$

4. Prove that, for $n \geq 2$, $\int |f|^2\chi_{\{|f|<n\}}d\mu = \int |f|^2\chi_{\{|f|<1\}}\, d\mu + \sum_{m=2}^{n}\int |f|^2\chi_{B_m}d\mu$.

5. Deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\int |f|^2\chi_{\{|f|<n\}}d\mu\right) < \infty$.

Exercise 9:
Let μ be a probability on a measurable space $(E, \mathcal{E})$, $\phi : (a, b) \longrightarrow \mathbb{R}$ be a convex function with $-\infty \leq a < b \leq +\infty$ and $f : E \longrightarrow \mathbb{R}$ be an $\mu-$integrable function.

1. Prove the *Jensen's Inequality* : $\phi\left(\int f d\mu\right) \leq \int \phi(f)d\mu$.
   Hint: Use the fact the graph of a convex function is the superior envelope (convex hull) of affine functions.

2. Write this inequality in terms of image probability $f_*\mu$.

3. Deduce the finite Jensen's inequality, obtained in the case: $E = (a, b)$, $\mathcal{E} = \mathcal{B}((a, b))$, $f(x) = x$ and $\mu = \sum_{i=1}^{n}\alpha_i\delta_{x_i}$ where δ_x is the Dirac measure at x and $\alpha_i \in \mathbb{R}$ such $\sum_{i=1}^{n}\alpha_i = 1$.