King Saud University
College of sciences
Department of mathematics
Second semester 1431/1432 H

Dr. Borhen Halouani

List of exercises $n°3$ (Math 580 Theory Measure I)

Exercise 1:
Let μ and ν be measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined as :

$$\mu = \sum_{n\geq 1} e^{-n}\delta_{1/n} \ ; \ \nu = \sum_{n\geq 1} e^{n}\delta_{1/n}.$$

1. Does the measures μ and ν are finite? probability measures? $\sigma-$finite? infinite?
2. Compute $\mu(\{0\})$, $\mu([0, \frac{1}{k}])$ for $k \geq 1$, $\lim\limits_{k\to\infty} \mu([0, \dfrac{1}{k}])$ and compare the results.
3. Compute $\nu(\{0\})$, $\nu([0, \frac{1}{k}])$ for $k \geq 1$, and compare the results.

Exercise 2:

1. Let $(X, \mathcal{A})$ be a measurable space and (μ_j) be a sequence of positives measures on $\mathcal{A}$. Assume that $\forall A \in \mathcal{A}$ and $\forall j \in \mathbb{N}$, $\mu_j(A) \leq \mu_{j+1}(A)$. For all $A \in \mathcal{A}$, we put $\mu(A) = \sup\limits_{j\in\mathbb{N}} \mu_j(A)$.
   Show that μ is a measure on $\mathcal{A}$.
2. On the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, we define for all $j \in \mathbb{N}$,
   $$\nu_j(A) = \text{card}(A \cap [j, +\infty]).$$
   Show that ν_j is a measure on $\mathcal{P}(\mathbb{N})$ and $\nu_j(A) \geq \nu_{j+1}(A)$.
3. Let $\nu : \mathcal{P}(\mathbb{N}) \longrightarrow [0, \infty]$ defined by $\nu(A) = \inf\limits_{j\in\mathbb{N}} \nu_j(A)$. Compute $\nu(\mathbb{N})$ and $\nu(\{k\})$ for $k \in \mathbb{N}$. Deduce that ν is not a measure.

Exercise 3:
Let λ be the Lebesgue measure on $\mathbb{R}$ and let $\varepsilon > 0$. Construct an open set Ω in $\mathbb{R}$ that is dense and $\lambda(\Omega) < \varepsilon$.
Exercise 4:
Let E be a nonempty set.

1. Let $\mathcal{E}$ be a $\sigma-$algebra on E and A be a subset of E. Show that the characteristic function χ_A is measurable if and only if $A \in \mathcal{E}$.

2. Let $\mathcal{A}$ be a partition at most countable of E, $\mathcal{E}$ be the $\sigma-$algebra generated by $\mathcal{A}$ and f be a real function on E. Show that f is measurable if and only if f is constant on each subset of $\mathcal{A}$.

3. Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) = \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$ is a Borelian function.

Exercise 5:
Let E be a Borelian set of $\mathbb{R}$ and $f : E \longrightarrow \mathbb{R}$ be a monotonic function. Show that f is a measurable function.

Exercise 6:
Let $(E, \mathcal{A})$ be a measurable space and $(f_n)_n$ be a sequence of measurable functions from E to $\mathbb{R}$. Show that the following sets:
$A = \left\{x \in E, \lim_{n\to\infty} f_n(x) = \infty\right\}$ and $B = \{x \in E, \text{the sequence } (f_n) \text{ is bounded }\}$ are measurable.

Exercise 7:
Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B})$ be a measurable space and $f : (X, \mathcal{A}) \longrightarrow (Y, \mathcal{B})$ be a measurable function. Show that the function $\mu_f : \mathcal{B} \longrightarrow [0, \infty]$ defined by $\mu_f(B) = \mu\left(f^{-1}(B)\right)$ is a measure on $\mathcal{B}$.

Exercise 8:(*Egoroff's theorem*)
Let $(X, \mathcal{A}, \mu)$ be a measure space such $\mu(X) < \infty$ and $f_n : (X, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a sequence of measurable functions.

1. Show that the set of convergence C of (f_n) is a measurable set.

2. We suppose that (f_n) is $\mu-$convergent to f a.e ($\mu(C^c) = 0$).
   For $k \in \mathbb{N}^*$, let $E_n^k = \bigcap_{i\geq n}\{|f_i - f| \leq \frac{1}{k}\}$. Show that $C \subset \bigcup_{n\geq 1} E_n^k$.
   Deduce that for all $\varepsilon > 0$, $\forall k \in \mathbb{N}^*$ there exists $n_{k,\varepsilon} \in \mathbb{N}^*$ such that
   $$\mu\left((E_{n_{k,\varepsilon}}^k)^c\right) < \frac{\varepsilon}{2^k}.$$

3. Deduce that $\forall\varepsilon > 0$ there exists $E_\varepsilon \in \mathcal{A}$ such that the sequence (f_n) converges uniformly to f on E_ε and $\mu\left(E_\varepsilon^c\right) < \varepsilon$.

4. Give a counterexample when $\mu(X) = +\infty$.