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Math 580 (Theory Measure I)

List of exercises $n°2$

Exercise 1:
Let X be an infinite countable set.
a) Show that the set of finite subsets of X is countable.
b) Deduce that the set of infinite subsets of X is uncountable.

Exercise 2:
a) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a monotonic function. Show that the set of all discontinuity points of f is a countable set.
Hint: Consider $J(n) = \left\{x \in (a, b] \backslash \mid f(x_+) - f(x_-) \mid > \frac{1}{n}\right\}$.
b) Same question with a monotonic function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Exercise 3:
Remember a real number is said *algebraic* if it is a root of a polynomial with integers coefficients.
Show that the set of all algebraic numbers is a countable set.

Exercise 4:
Let X be a nonempty set. Let $(A_n)_{n\geq 1}$ be a sequence of subsets of X.
a) Show that

$$\chi_{\bigcup_{i=1}^{n} A_i} = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I\subset\{1,2,\ldots,n\},|I|=k} \chi_{\bigcap_{i\in I} A_i}.$$

b) If X is finite, deduce Poincare's formula:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I\subset\{1,2,\ldots,n\},|I|=k} \left|\bigcap_{i\in I} A_i\right|.$$

Exercise 5:
Let $\mathfrak{A} = \{A, B, C\}$ be a partition of X on three subsets. Describe the $\sigma-$algebra generated by $\mathfrak{A}$.

Exercise 6:
Let $\mathfrak{A}$ and $\mathfrak{F}$ be two $\sigma-$algebras of X. Describe the $\sigma-$algebras generated by $\mathfrak{A} \bigcap \mathfrak{F}$ and by $\mathfrak{A} \bigcup \mathfrak{F}$.

Exercise 7:
Let $\mathfrak{A}$ be an algebra and μ is a measure defined on it. Let $A, B \in \mathfrak{A}$. Prove that

$$|\mu(A) - \mu(B)| \leq \mu(A \Delta B)$$

Hint: $A \subset A \bigcup (A \Delta B) = A \bigcup B$.
Check whether the inequality holds if μ is an outer measure.

Exercise 8:

Let X be a set, $\mathfrak{A}$ an algebra of its subsets. Let $\widetilde{\mathfrak{A}}$ be the $\sigma-$algebra of Caratheodory measurable subsets of X. Suppose that $A \subset X$ is such that for any $\varepsilon > 0$ there exists $A_\varepsilon \in \mathfrak{A}$ such that $\mu^*(A \Delta A_\varepsilon) < \varepsilon$. Prove that $A \in \widetilde{\mathfrak{A}}$.

Exercise 9:

For all $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. Define $x.A = \{x.a \backslash a \in A\}$ and

$$\mu^*(A) = inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n), A \subset \bigcup_{n=1}^{\infty} (a_n, b_n); (a_n, b_n) \texttt{finite intervals} \right\}$$

a) Show that μ^* is an outer measure on $\mathcal{P}(\mathbb{R})$.
b) Show that $\mu^*(x.A) = |x|\mu^*(A)$.
c) Let $c \in \mathbb{R}$. Define $A_1 = \{a \in A, a < c\}$ and $A_2 = \{a \in A, a \geq c\}$. Show that

$$\mu^*(A) = \mu^*(A_1) + \mu^*(A_2).$$