Q1

$* X \sim unif(0,1) \Rightarrow f_X(x) = \frac{1}{1} = 1 \;,\; 0<x<1$

$* Y = -2\ln X \Rightarrow \underset{\sim\sim\sim}{X = e^{-\frac{y}{2}}} \Rightarrow \frac{d}{dy}X = -\frac{1}{2}e^{-\frac{y}{2}} \Rightarrow \underset{\sim\sim\sim}{\left|\frac{d}{dy}X\right| = \frac{1}{2}e^{-\frac{y}{2}}}$

$* 0<X<1 \Rightarrow \ln 0 < \ln x < \ln 1 \Rightarrow -\infty < \ln x < 0 \Rightarrow \infty > -2\ln x > 0 \Rightarrow \underset{\sim\sim\sim}{0<y<\infty}$

$* f_Y(y) = f_X(x)\left|\frac{d}{dy}x\right| = \frac{1}{2}e^{-\frac{y}{2}}$

$\therefore Y \sim exp(\lambda = \frac{1}{2})$

Q2

* $X \sim N(\mu, \sigma^2) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$

* $Y = e^X \Rightarrow X = \ln Y \Rightarrow \frac{d}{dy} X = \frac{1}{y} \Rightarrow \left|\frac{d}{dy} X\right| = \frac{1}{y}$

* $-\infty < x < \infty \Rightarrow 0 < e^x < \infty \Rightarrow 0 < y < \infty$

* $f_Y(y) = f_X(x) \left|\frac{d}{dy} x\right| = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2} \frac{1}{y}$

Q3

* $X \sim \exp(1) \Rightarrow f_X(x) = e^{-x}, \quad \infty > x > 0$

* $Y = -\ln X \Rightarrow X = e^{-Y} \Rightarrow \frac{d}{dy} X = -e^{-Y} \Rightarrow \left|\frac{d}{dy} X\right| = e^{-Y}$

* $x > 0 \Rightarrow \infty > \ln x > -\infty \Rightarrow -\infty < -\ln x < \infty \Rightarrow -\infty < y < \infty$

* $f_Y(y) = f_X(x) \left|\frac{d}{dy} x\right| = e^{-e^{-y}} e^{-y} = e^{-(y + e^{-y})}$

Q4

* $X \sim \text{unif}(0,1) \Rightarrow f_X(x) = 1, \quad 0 < x < 1$

* $Y = \sqrt{X} \Rightarrow X = Y^2 \Rightarrow \frac{d}{dy} X = 2Y \Rightarrow \left|\frac{d}{dy} X\right| = 2Y$

* $0 < x < 1 \Rightarrow 0 < \sqrt{x} < 1 \Rightarrow 0 < y < 1$

* $f_Y(y) = f_X(x) \left|\frac{d}{dy} x\right| = 2y$

Q5

* $X \sim$ dis. $f_X(x) = \frac{1}{2}x, \quad 0<x<2$

* $Y = X^3 \Rightarrow \underline{X = Y^{\frac{1}{3}}} \Rightarrow \frac{d}{dy}X = \frac{1}{3}Y^{-\frac{2}{3}} \Rightarrow \underline{\left|\frac{d}{dy}X\right| = \frac{1}{3}Y^{-\frac{2}{3}}}$

* $0<x<2 \Rightarrow 0<x^3<8 \Rightarrow \underline{0<y<8}$

* $f_Y(y) = f_X(x)\left|\frac{d}{dy}x\right| = \frac{1}{2}y^{\frac{1}{3}} \frac{1}{3}y^{-\frac{2}{3}} = \frac{1}{6}y^{-\frac{1}{3}}$

* $P(\frac{1}{2}<X<1) = \frac{1}{2}\int_{1/2}^{1} x\,dx = \frac{1}{4}x^2\Big|_{1/2}^{1} = \frac{1}{4}(-\frac{1}{4}+1) = \frac{1}{4}(\frac{3}{4}) = \frac{3}{16}$

$P(\frac{1}{8}<Y<1) = \frac{1}{6}\int_{1/8}^{1} y^{-\frac{1}{3}}dy = \frac{1}{6}(\frac{3}{2})y^{2/3}\Big|_{1/8}^{1} = \frac{1}{4}(1-\frac{1}{4}) = \frac{1}{4}(\frac{3}{4}) = \frac{3}{16}$

we can see that are the same because

$\frac{1}{2}<X<1 \Rightarrow \frac{1}{8}<X^3<1 \Rightarrow \frac{1}{8}<Y<1$ and so $P(\frac{1}{2}<X<1) = P(\frac{1}{8}<Y<1)$

Q6

$$\left.\begin{array}{l} * X \sim unif(0,1) \Rightarrow f(x)=1, \quad 0<x<1 \\ Y \sim exp(1) \Rightarrow f(y) = e^{-y}, \quad y>0 \end{array}\right\} f(x,y) = e^{-y}, \quad 0<x<1, \ y>0$$

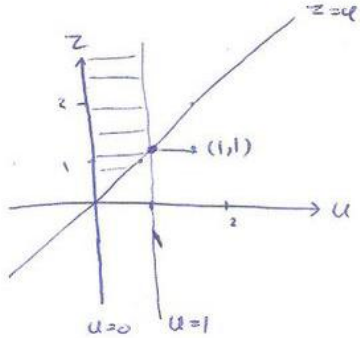
* $Z = X+Y \Rightarrow Y = Z-X$

$\underline{U = X}$

$\therefore \underline{Y = Z-U}$

*
$$\left.\begin{array}{l} 0<x<1 \\ 0<y<\infty \end{array}\right\} \Rightarrow \begin{array}{l} 0<x+y<\infty \Rightarrow \underline{0<z<\infty} \\ 0<x<1 \Rightarrow \underline{0<u<1} \end{array}$$

$$\left.\begin{array}{l} 0<x<1 \\ 0<y<\infty \end{array}\right\} \Rightarrow \begin{array}{l} 0<u<1 \\ 0<z-u<\infty \Rightarrow \underline{u<z<\infty} \end{array}$$

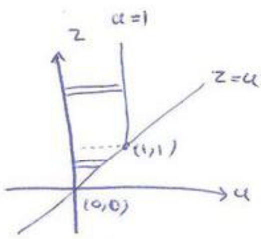


* $J(x,y) = \begin{vmatrix} \frac{\partial}{\partial x}z & \frac{\partial}{\partial y}z \\ \frac{\partial}{\partial x}u & \frac{\partial}{\partial y}u \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0-1 = -1 \Rightarrow |J(x,y)|^{-1} = 1$

* $f(z,u) = f(x,y)\,|J(x,y)|^{-1} = e^{-(z-u)}$

* $f(z) = \int f(z,u)\,du = \begin{cases} \int_0^z e^{-(z-u)}du, & 0<z<1 \\ \int_0^1 e^{-(z-u)}du, & 1<z<\infty \end{cases}$

$= \begin{cases} 1-e^{-z}, & 0<z<1 \\ (e-1)e^{-z}, & 1<z<\infty \end{cases}$



Q7 $* X_1, X_2 \sim \exp(\lambda) \Rightarrow f(x_1) = \lambda e^{-\lambda x_1}, \; x_1 > 0$

$f(x_2) = \lambda e^{-\lambda x_2}, \; x_2 > 0$

X_1, X_2 indep. $\therefore f(x_1, x_2) = f(x_1) f(x_2)$

$= \lambda^2 e^{-\lambda(x_1 + x_2)}, \; x_1 > 0, \; x_2 > 0$

$* \; Y_1 = X_1 + X_2 \Rightarrow X_2 = Y_1 - X_1$

$Y_2 = e^{X_1} \Rightarrow \underline{X_1 = \ln Y_2}$

$\therefore X_2 = Y_1 - X_1 = Y_1 - \ln Y_2 \Rightarrow \underline{X_2 = Y_1 - \ln Y_2}$

$* \; \left.\begin{array}{l} 0 < x_1 < \infty \\ 0 < x_2 < \infty \end{array}\right\} \Rightarrow \begin{array}{l} 0 < x_1 + x_2 < \infty \Rightarrow \underline{0 < y_1 < \infty} \\ 0 < e^{x_2} < \infty \Rightarrow \underline{0 < y_2 < \infty} \end{array}$

$\left.\begin{array}{l} 0 < x_1 < \infty \\ 0 < x_2 < \infty \end{array}\right\} \Rightarrow \begin{array}{l} \underline{0 < \ln y_2 < \infty} \\ 0 < y_1 - \ln y_2 < \infty \Rightarrow \underline{\ln y_2 < y_1 < \infty} \end{array}$

* $J(x_1,x_2) = \begin{vmatrix} \frac{\partial}{\partial x_1} y_1 & \frac{\partial}{\partial x_2} y_1 \\ \frac{\partial}{\partial x_1} y_2 & \frac{\partial}{\partial x_2} y_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ e^{x_1} & 0 \end{vmatrix} = 0 - e^{x_1} = -e^{x_1} = -e^{\ln y_2} = -y_2$

$\Rightarrow \quad |J(x_1,x_2)|^{-1} = \frac{1}{y_2}$

* $f(y_1,y_2) = f(x_1,x_2)\,|J(x_1,x_2)|^{-1} = \lambda^2 e^{-\lambda(\ln y_2 + y_1 - \ln y_2)} \frac{1}{y_2}$

$= \lambda^2 e^{-\lambda y_1} \frac{1}{y_2}$

Q8 a)

* $X_1 \sim exp(\lambda_1) \Rightarrow f(x_1) = \lambda_1 e^{-\lambda_1 x_1}, \; x_1 > 0$
$X_2 \sim exp(\lambda_2) \quad f(x_2) = \lambda_2 e^{-\lambda_2 x_2}, \; x_2 > 0$

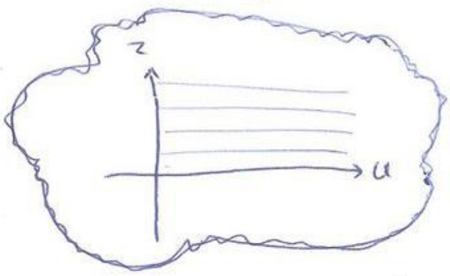
X_1, X_2 indep.
$f(x_1,x_2) = f(x_1) f(x_2)$
$= \lambda_1 \lambda_2 e^{-(\lambda_1 x_1 + \lambda_2 x_2)}, \; x_1 > 0, \; x_2 > 0$

* $Z = \frac{X_1}{X_2} \Rightarrow X_2 = \frac{X_1}{Z} = \frac{U}{Z} \Rightarrow \underline{X_2 = \frac{U}{Z}}$

$U = X_1 \Rightarrow \underline{X_1 = U}$

* $\left.\begin{matrix} x_1 > 0 \\ x_2 > 0 \end{matrix}\right\} \Rightarrow \begin{matrix} u > 0 \\ \underline{z > 0} \end{matrix}$

$\left.\begin{matrix} x_1 > 0 \\ x_2 > 0 \end{matrix}\right\} \Rightarrow \begin{matrix} u > 0 \\ \frac{u}{z} > 0 \Rightarrow u > 0, z > 0 \text{ or } u < 0, z < 0 \end{matrix}$



* $J(x_1,x_2) = \begin{vmatrix} \frac{\partial}{\partial x_1} z & \frac{\partial}{\partial x_2} z \\ \frac{\partial}{\partial x_1} u & \frac{\partial}{\partial x_2} u \end{vmatrix} = \begin{vmatrix} \frac{1}{x_2} & \frac{-x_1}{x_2^2} \\ 1 & 0 \end{vmatrix} = \frac{+x_1}{x_2^2} = \frac{u}{(u/z)^2} = \frac{z^2}{u}$

$\Rightarrow \quad |J(x_1,x_2)|^{-1} = \frac{u}{z^2}$

* $f(z,u) = f(x_1,x_2)\,|J(x_1,x_2)|^{-1} = \lambda_1 \lambda_2 e^{-(\lambda_1 u + \lambda_2 \frac{u}{z})} \frac{u}{z^2} = \lambda_1 \lambda_2 \frac{u}{z^2} e^{-u(\lambda_1 + \lambda_2 \frac{1}{z})}$

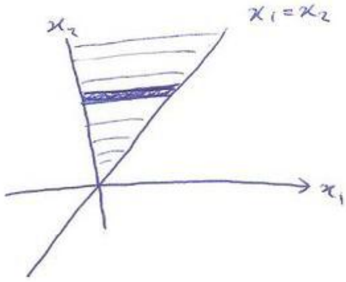
$$* \; f(z) = \int_0^\infty f(z,u)\,du = \lambda_1\lambda_2 \frac{1}{z^2} \int_0^\infty u\, e^{-u(\lambda_1 + \lambda_2 \frac{1}{z})}\,du$$

$$= \lambda_1\lambda_2 \frac{1}{z^2} \frac{\Gamma 2}{(\lambda_1+\lambda_2\frac{1}{z})^2} = \lambda_1\lambda_2\frac{1}{z^2}\frac{1}{\left(\frac{\lambda_1 z+\lambda_2}{z}\right)^2} = \frac{\lambda_1\lambda_2}{(\lambda_1 z+\lambda_2)^2}, \quad z>0$$

$$\Rightarrow F(z) = \begin{cases} 0, & z<0 \\ \int_0^z \frac{\lambda_1\lambda_2}{(t\lambda_1+\lambda_2)^2}\,dt = \lambda_1\lambda_2\int_0^z (t\lambda_1+\lambda_2)^{-2}\,dt & \\ = \lambda_1\lambda_2 \frac{(t\lambda_1+\lambda_2)^{-1}}{-\lambda_1}\Big|_0^z = 1-\frac{\lambda_2}{z\lambda_1+\lambda_2} = \frac{z\lambda_1}{z\lambda_1+\lambda_2}, & z>0 \end{cases}$$

b)

$$P(X_1<X_2) = \int_0^\infty\int_0^{x_2} f(x_1,x_2)\,dx_1\,dx_2$$



$$= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2}\left[-e^{-\lambda_1 x_1}\Big|_0^{x_2}\right]dx_2$$

$$= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2}(1-e^{-\lambda_1 x_2})\,dx_2$$

$$= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2}\,dx_2 - \frac{\lambda_2}{\lambda_1+\lambda_2}\int_0^\infty (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x_2}\,dx_2$$

$$= 1-\frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

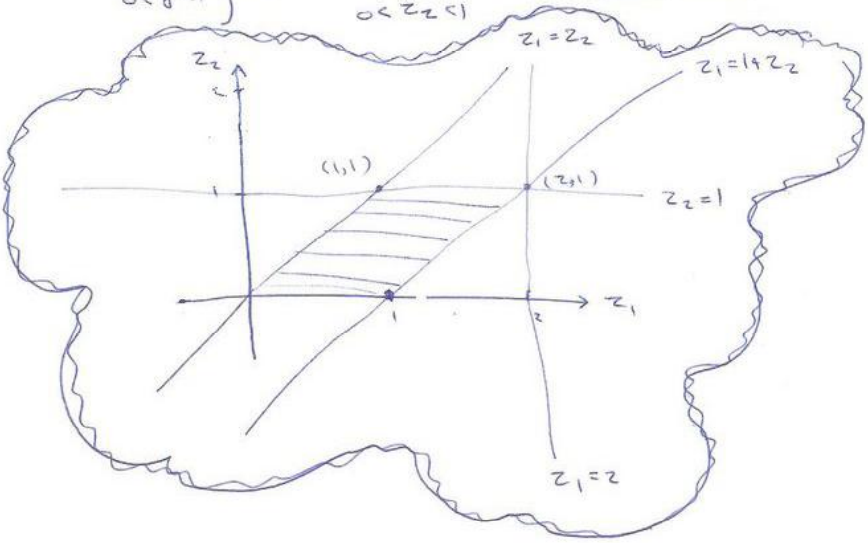
Q9 a)

* $X, Y \sim \text{unif}(0,1) \Rightarrow f(x) = 1, \quad 0<x<1$
  $f(y)=1, \quad 0<y<1$ $\Big\}$ X, Y indep.
  $f(x,y) = f(x) f(y) = 1, \quad 0<x<1, \; 0<y<1$

* $Z_1 = X+Y \Rightarrow X = Z_1 - Y \Rightarrow \underline{X = Z_1 - Z_2}$
  $\underline{Z_2 = Y}$

* $\left.\begin{array}{l} 0<x<1 \\ 0<y<1 \end{array}\right\} \Rightarrow \begin{array}{l} 0<x+y<2 \Rightarrow \underline{0<z_1<2} \\ 0<y<1 \Rightarrow \underline{0<z_2<1} \end{array}$

  $\left.\begin{array}{l} 0<x<1 \\ 0<y<1 \end{array}\right\} \Rightarrow \begin{array}{l} 0<z_1-z_2<1 \Rightarrow \underline{z_2<z_1<1+z_2} \\ 0<z_2<1 \end{array}$



* $J(x,y) = \begin{vmatrix} \frac{\partial}{\partial x} z_1 & \frac{\partial}{\partial y} z_1 \\ \frac{\partial}{\partial x} z_2 & \frac{\partial}{\partial y} z_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \Rightarrow |J(x,y)|^{-1} = 1$

* $f(z_1, z_2) = f(x,y) \, |J(x,y)|^{-1} = 1$

b) $f(z_1) = \int f(z_1, z_2) \, dz_2 = \begin{cases} \int_0^{z_1} 1 \, dz_2 = z_1, & 0<z_1<1 \\ \int_{z_1-1}^{1} 1 \, dz_2 = 2 - z_1, & 1<z<2 \end{cases}$

Q10 a)

$*X, Y \sim exp(1)$ which they are indep. $\Big\}$ $f(x,y) = f(x) f(y) = e^{-x} e^{-y} = e^{-(x+y)}$, $x>0, y>0$

$* \; Z = X+Y \Rightarrow X = Z-Y$

$U = \frac{X}{X+Y} \Rightarrow U = \frac{X}{Z} = \frac{Z-Y}{Z} \Rightarrow UZ = Z-Y \Rightarrow Y = Z - ZU = Z(1-U) \Rightarrow \underline{Y = Z(1-U)}$

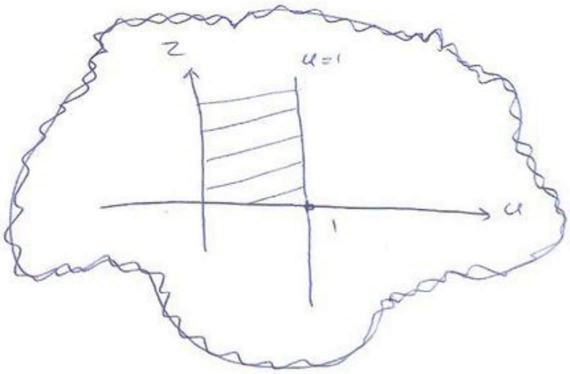
$\therefore X = Z - Y = Z - Z(1-U) = ZU \Rightarrow \underline{X = ZU}$

$*$ $\left.\begin{matrix} x>0 \\ y>0 \end{matrix}\right\} \Rightarrow$ $x+y>0 \Rightarrow \underline{z>0}$

$\frac{x}{x+y} > \frac{0}{0}$ (x), but $\frac{x}{x+y} > 0 \Rightarrow \underline{u>0}$

$\left.\begin{matrix} x>0 \\ y>0 \end{matrix}\right\} \Rightarrow$ $zu>0 \Rightarrow z>0$ or $u>0$

$z(1-u)>0 \Rightarrow z>0$ or $(1-u)>0 \Rightarrow z>0$ or $\underline{1>u}$



$* \; J(x,y) = \begin{vmatrix} \frac{\partial}{\partial x} z & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} u & \frac{\partial}{\partial y} u \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{(x+y)-x}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{vmatrix} = \frac{-1}{(x+y)} = \frac{-1}{z}$

$\Rightarrow |J(x,y)|^{-1} = z$

$* \; f(z,u) = f(x,y) \, |J(x,y)|^{-1} = e^{-(zu + z(1-u))} (z) = z e^{-z}$

b) pdf of u:

$f(y) = \int_0^\infty z e^{-z} \, dz = \frac{\Gamma 2}{1^2} = 1$, $0<u<1$

Q11 a)

* $X \sim Gama(\alpha, \lambda)$

$Y \sim Gama(\alpha, \lambda)$

$\therefore f(x) = \frac{\lambda^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\lambda x}, \quad x>0$

$f(y) = \frac{\lambda^\alpha}{\Gamma\alpha} y^{\alpha-1} e^{-\lambda y}, \quad y>0$

as X, Y are indep. so,

$f(x,y) = f(x) f(y)$

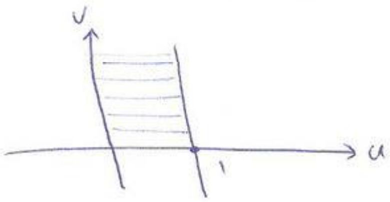
$= \left(\frac{\lambda^\alpha}{\Gamma\alpha}\right)^2 x^{\alpha-1} y^{\alpha-1} e^{-\lambda(x+y)}, \quad x>0, y>0$

From (19) we can see that:

* $X = VU$

$Y = V(1-U)$

* $V>0, \quad U>0, \quad 1>U$



* $|J(x,y)|^{-1} = V$

but:

* $f(V,U) = f(x,y)\,|J(x,y)|^{-1} = \left(\frac{\lambda^\alpha}{\Gamma\alpha}\right)^2 (VU)^{\alpha-1} (V(1-U))^{\alpha-1} e^{-\lambda(VU+V(1-U))} (V)$

$= \left(\frac{\lambda^\alpha}{\Gamma\alpha}\right)^2 V^{2\alpha-1} U^{\alpha-1} (1-U)^{\alpha-1} e^{-\lambda(V)}$

b) $f(U) = \int f(V,U)\, dV = \left(\frac{\lambda^\alpha}{\Gamma\alpha}\right)^2 U^{\alpha-1} (1-U)^{\alpha-1} \int_0^\infty V^{2\alpha-1} e^{-\lambda V} dV$

$= \left(\frac{\lambda^\alpha}{\Gamma\alpha}\right)^2 U^{\alpha-1}(1-U)^{\alpha-1} \frac{\Gamma 2\alpha}{\lambda^{2\alpha}} \quad \left(\text{From } \int_0^\infty x^{n-1} e^{-kx} dx = \frac{\Gamma n}{k^n}\right)$

$= \frac{1}{(\Gamma\alpha)^2} \frac{\Gamma 2\alpha}{1} (U)^{\alpha-1} (1-U)^{\alpha-1}$

$= \frac{\Gamma 2\alpha}{(\Gamma\alpha)^2} U^{\alpha-1} (1-U)^{\alpha-1} = \frac{\Gamma(\alpha+\alpha)}{\Gamma\alpha\,\Gamma\alpha} U^{\alpha-1}(1-U)^{\alpha-1}, \quad 0<U<1$

$\left(X \sim Beta(a,b) \Rightarrow f(x) = \frac{\Gamma a+b}{\Gamma a \Gamma b} x^{a-1} (1-x)^{b-1}, \quad 0<x<1 \right)$

$\therefore U \sim Beta(\alpha, \alpha)$