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Characterization of quantale-valued metric spaces and quantale-valued partial metric spaces by convergence

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ABSTRACT

We identify two categories of quantale-valued convergence tower spaces that are isomorphic to the categories of quantale-valued metric spaces and quantale-valued partial metric spaces, respectively. As an application we state a quantale-valued metrization theorem for quantale-valued convergence tower groups.

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KEYWORDS: *L-metric space; L-partial metric space; L-convergence tower space; L-convergence tower group; metrization.*

1. INTRODUCTION

There are different generalizations of metric spaces. One of them solves the problem of assigning a precise value to the distance between two points by allowing instead the assignment of a probability distribution for each pair of points, the value of which at $u \in [0, \infty]$ giving the probabilty that the distance between the points is less than u. A thorough treatment of these *probabilistic metric spaces* can be found in [28]. From a different perspective, metric spaces are viewed as categories in [19] and later, in [9] it has been shown that not only

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classical metric spaces but also probabilistic metric spaces are special instances of this approach. The main idea here is to replace the interval $[0, \infty]$ as the codomain of a metric by a quantale. For this reason, we speak of *quantale-valued metric spaces*. Another generalization of metric spaces deals with the problem, that self-distances are not always zero. Relaxing this requirement and at the same time enforcing the transivity axiom leads to the theory of *partial metric spaces* [17, 21]. This concept was independently, and in much wider generality, introduced under the name *M-valued set* by Höhle [11], where the relationship with the general view point of [19] becomes obvious, as also M-valued sets take their values in a quantale.

All these generalizations allow the introduction of underlying topological spaces and, in consequence, of a concept of convergence. In this paper, we look at convergence for quantale-valued metric spaces from a different perspective. Rather than describing a concept of convergence *underlying* a quantale-valued metric space, we are looking for a concept of convergence that *characterizes* such spaces. The key point is here to allow different grades of convergence, where these grades are interpreted as values in the quantale. In this sense, a filter in a quantale-valued (partial) metric space converges to a point with a certain grade. We obtain in this way a family of convergence structures on a set indexed by the quantale. For the unit interval as "index set" spaces with such *towers of convergence structures* were first studied by Richardson and Kent under the name *probabilistic convergence spaces* [26]. In a more general setting, quantale-valued convergence towers are considered in [15]. In this paper, we identify a set of axioms, such that the quantale-valued (partial) convergence tower spaces satisfying these axioms can be identified with quantale-valued (partial) metric spaces.

The paper is organized as follows. In the second section, we collect the necessary concepts and notions from lattice theory and fix the notation. In the third section, we study quantale-valued metric spaces and quantale-valued convergence tower spaces and their relationship. The fourth section then states an axiom which ensures the isomorphy of quantale-valued metric spaces and a category of quantale-valued convergence tower spaces satisfying this axiom. In a similar fashion, in Section 5, it is shown that quantale-valued partial convergence tower spaces satisfying certain axioms can be used to characterize quantale-valued partial metric spaces. Finally, in Section 6, we apply our results and state a quantale-valued metrization theorem for quantale-valued convergence tower groups.

2. Preliminaries

Let L be a complete lattice. We assume that L is non-trivial in the sense that $\top \neq \bot$ for the top element $\top$ and the bottom element $\bot$. In any complete lattice L we can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$ and $\alpha \lhd \bigvee_{j \in J} \beta_j$ iff $\alpha \lhd \beta_i$ for some $i \in J$. A complete lattice is *completely* *distributive* (sometimes called *constructively completely distributive*) if and only if we have $\alpha = \bigvee\{\beta \ : \ \beta \lhd \alpha\}$ for any $\alpha \in L$, [25]. (For a more accessible proof of the equivalence of this condition with the classical concept of complete distributivity in the presence of the Axiom of Choice see e.g. Theorem 7.2.3 in [1].) In a completely distributive lattice L, from $\alpha \lhd \beta = \bigvee\{\gamma \in L \ : \ \gamma \lhd \beta\}$ we infer the existence of $\gamma \in L$ such that $\alpha \lhd \gamma \lhd \beta$, i.e. L satisfies the so-called *interpolation property*. For more results on lattices we refer to [10].

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice, is called a *quantale* [27] if $(L, *)$ is a semigroup, and $*$ is distributive over arbitrary joins, i.e.

$$\left(\bigvee_{i\in J} \alpha_i\right) * \beta = \bigvee_{i\in J} (\alpha_i * \beta) \quad \text{and} \quad \beta * \left(\bigvee_{i\in J} \alpha_i\right) = \bigvee_{i\in J} (\beta * \alpha_i).$$

A quantale $\mathsf{L} = (L, \leq, *)$ is called *commutative* if $(L, *)$ is a commutative semigroup and it is called *integral* if the top element of L acts as the unit, i.e. if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha \in L$. A quantale $\mathsf{L} = (L, \leq, *)$ is called an *MV-algebra* [13], if for all $\alpha, \beta \in L$ we have $(\alpha \to \beta) \to \beta = \alpha \vee \beta$. In a quantale we can define an implication operator by

$$\alpha \to \beta = \bigvee\{\gamma \in L \ : \ \alpha * \gamma \leq \beta\}.$$

Then $\delta \leq \alpha \to \beta$ if and only if $\delta * \alpha \leq \beta$.

We consider in this paper only commutative and integral quantales $\mathsf{L} = (L, \leq, *)$ with completely distributive lattices L.

Example 2.1. A *triangular norm* or *t-norm* is a binary operation $*$ on the unit interval $[0, 1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. The triple $\mathsf{L} = ([0, 1], \leq, *)$ can be considered as a quantale if the t-norm is left-continuous. The three most commonly used (left-continuous) t-norms are:

- the minimum t-norm: $\alpha * \beta = \alpha \wedge \beta$,
- the product t-norm: $\alpha * \beta = \alpha \cdot \beta$,
- the Lukasiewicz t-norm: $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.

For the minimum t-norm we obtain $\alpha \to \beta = \begin{cases} \top & \text{if } \alpha \leq \beta \\ \beta & \text{if } \alpha > \beta \end{cases}$. For the product t-norm we have $\alpha \to \beta = \frac{\beta}{\alpha} \wedge 1$ and for the Lukasiewicz t-norm we have $\alpha \to \beta = (1 - \alpha + \beta) \wedge 1$.

Example 2.2. [19] The interval $[0, \infty]$ with the opposite order and addition as the quantale operation $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + a = \infty$ for all $\alpha, \beta \in [0, \infty]$) is a quantale $\mathsf{L} = ([0, \infty], \geq, +)$, see e.g. [9]. We have here $\alpha \to \beta = (\beta - \alpha) \vee 0$.

Example 2.3. A function $\varphi : [0, \infty] \longrightarrow [0, 1]$, which is non-decreasing, left-continuous on $(0, \infty)$ – in the sense that for all $x \in (0, \infty)$ we have $\varphi(x) = \sup_{z<x} \varphi(z)$ – and satisfies $\varphi(0) = 0$ is called a *distance distribution function*

[28]. The set of all distance distribution functions is denoted by Δ^+. For example, for each $0 \le a \le \infty$ the functions

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if } 0 \le x \le a \\ 1 & \text{if } a < x \le \infty \end{cases}$$

are in Δ^+. The set Δ^+ is ordered pointwise, i.e. for $\varphi, \psi \in \Delta^+$ we define $\varphi \le \psi$ if for all $x \ge 0$ we have $\varphi(x) \le \psi(x)$. The bottom element of Δ^+ is then ε_∞ and the top element is ε_0. The set Δ^+ with this order then becomes a complete lattice. We note that $\bigwedge_{i\in I} \varphi_i$ is in general not the pointwise infimum. It is not difficult to show that for all $x \in [0,\infty]$ we have $\bigwedge_{j\in J} \varphi_j(x) = \sup_{z<x} \inf_{j\in J} \varphi_j(z)$ with the pointwise infimum $\inf_{j\in J} \varphi_j$. It is shown in [9] that Δ^+ is completely distributive.

A binary operation, $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, which is commutative, associative, non-decreasing in each place and that satisfies the boundary condition $\varphi * \varepsilon_0 = \varphi$ for all $\varphi \in \Delta^+$, is called a *triangle function* [28]. A triangle function is called *sup-continuous* [28], if $(\bigvee_{i\in I} \varphi_i) * \psi = \bigvee_{i\in I} (\varphi_i * \psi)$ for all $\varphi_i, \psi \in \Delta^+$, $(i \in I)$, i.e. if $\mathsf{L} = (\Delta^+, \le, *)$ is a quantale.

For a set X, we denote its power set by $P(X)$ and the set of all filters $\mathbb{F}, \mathbb{G}, ...$ on X by $\mathbb{F}(X)$. The set $\mathbb{F}(X)$ is ordered by set inclusion and maximal elements of $\mathbb{F}(X)$ in this order are called *ultrafilters*. The set of all ultrafilters on X is denoted by $\mathbb{U}(X)$. In particular, for each $x \in X$, the point filter $[x] = \{A \subseteq X : x \in A\} \in \mathbb{F}(X)$ is an ultrafilter. If $\mathbb{F} \in \mathbb{F}(X)$ and $f : X \longrightarrow Y$ is a mapping, then we define $f(\mathbb{F}) \in \mathbb{F}(Y)$ by $f(\mathbb{F}) = \{G \subseteq Y : f(F) \subseteq G \text{ for some } F \in \mathbb{F}\}$.

We assume some familiarity with category theory and refer to the textbooks [2] and [23] for more details and notation.

3. L-metric spaces as L-convergence tower spaces

For a quantale $\mathsf{L} = (L, \le, *)$, an L*-metric space* [19, 9] is a pair (X, d) of a set X and a mapping $d : X \times X \longrightarrow L$ such that

(LM1) $d(x,x) = \top$ for all $x \in X$ (*reflexivity*);
(LM2) $d(x,y) * d(y,z) \le d(x,z)$ for all $x,y,z \in X$ (*transitivity*).

A mapping between two L-metric spaces, $f : (X,d) \longrightarrow (X',d')$ is called an L-*metric morphism* if $d(x_1,x_2) \le d'(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. We denote the category of L-metric spaces with L-metric morphisms by L-MET.

In case $L = \{0,1\}$, an L-metric space is a preordered set. If $\mathsf{L} = ([0,\infty], \ge, +)$, an L-metric space is a quasimetric space. If $\mathsf{L} = (\Delta^+, \le, *)$, an L-metric space is a probabilistic quasimetric space, see [9].

Let X be a set. A family of mappings $\overline{c} = (c_\alpha : \mathbb{F}(X) \longrightarrow P(X))_{\alpha\in L}$ which satisfies the axioms

(LC1) $x \in c_\alpha([x])$ for all $x \in X, \alpha \in L$;
(LC2) $c_\alpha(\mathbb{F}) \subseteq c_\alpha(\mathbb{G})$ whenever $\mathbb{F} \le \mathbb{G}$;
(LC3) $c_\beta(\mathbb{F}) \subseteq c_\alpha(\mathbb{F})$ whenever $\alpha \le \beta$;
(LC4) $x \in c_\bot(\mathbb{F})$ for all $x \in X, \mathbb{F} \in \mathbb{F}(X)$;

is called an L-*convergence tower on* X and the pair $(X,\overline{c})$ is called an L-*convergence tower space*. A mapping $f: X \longrightarrow X'$ between the L-convergence tower spaces $(X,\overline{c})$ and $(X',\overline{c'})$, is called *continuous* if, for all $x \in X$ and all $\mathbb{F} \in \mathbb{F}(X)$, $f(x) \in c'_\alpha(f(\mathbb{F}))$ whenever $x \in c_\alpha(\mathbb{F})$. The category of L-convergence tower spaces with continuous mappings as morphisms is denoted by $\mathsf{L\text{-}CTS}$. We note that an L-convergence tower space is a stratified $\{0,1\}\{0,1\}L$-convergence tower space in the definition of [15].

For $L = \{0,1\}$ an L-convergence tower space is a generalized convergence space [24], for $\mathsf{L} = ([0,1],\leq,\wedge)$ we obtain the probabilistic convergence spaces in the definition of Richardson and Kent [26] and for $\mathsf{L} = (\Delta^+,\leq,*)$ we obtain the probabilistic convergence spaces in [14]. For $\mathsf{L} = ([0,\infty],\geq,+)$ we obtain - demanding one additional axiom - the limit tower spaces of Brock and Kent [8]. It follows from [15] that the category $\mathsf{L\text{-}CTS}$ is topological, Cartesian closed and extensional.

An L-convergence tower space $(X,\overline{c})$ is called *pretopological* if the axiom

(LCPT) $\bigcap_{i\in I} c_\alpha(\mathbb{F}_i) \subseteq c_\alpha(\bigwedge_{i\in I} \mathbb{F}_i)$ whenever $\alpha \in L$ and $(\mathbb{F}_i)_{i\in I} \in \mathbb{F}(X)^I$

is satisfied. It is called *left-continuous* if for all subsets $M \subseteq L$ we have

(LCLC) $x \in c_{\bigvee M}(\mathbb{F})$ whenever $x \in c_\alpha(\mathbb{F})$ for all $\alpha \in M$.

It is called $*$-*transitive* if

(LCT) $x \in c_{\alpha*\beta}([z])$ whenever $x \in c_\alpha([y])$ and $y \in c_\beta([z])$.

Remark 3.1. A left-continuous L-convergence tower $\overline{c}$ can be identified with a *limit function* $\lambda : \mathbb{F}(X) \longrightarrow L^X$, where $\lambda(\mathbb{F})(x) = \bigvee\{\alpha \in L : x \in c_\alpha(\mathbb{F})\}$.

We call a left-continuous, $*$-transitive and pretopological L-convergence tower space an L-*premetric convergence tower space* and denote the category of these spaces spaces by $\mathsf{L\text{-}PreMET\text{-}CTS}$.

Proposition 3.2. *Let* $(X,d) \in |\mathsf{L\text{-}MET}|$. *Define*

$$x \in c^d_\alpha(\mathbb{F}) \iff \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y) \geq \alpha.$$

Then $(X,\overline{c^d}) \in |\mathsf{L\text{-}CTS}|$.

Proof. (LC1) We have $\bigvee_{F\in[x]} \bigwedge_{y\in F} d(x,y) \geq d(x,x) = \top \geq \alpha$ (with $\{x\} \in [x]$), and hence $x \in c^d_\alpha([x])$ for all $\alpha \in L$ and all $x \in X$.

(LC2), (LC3) and (LC4) are obvious. □

Proposition 3.3. *Let* $(X,d) \in |\mathsf{L\text{-}MET}|$. *Then* $(X,\overline{c^d})$ *is* $*$*-transitive, left-continuous and pretopological.*

Proof. For transitivity, we first note that $\bigvee_{F\in[y]} \bigwedge_{a\in F} d(x,a) \leq \bigvee_{F\in[y]} d(x,y) = d(x,y)$ and, using $F = \{y\}$ we also have $\bigvee_{F\in[y]} \bigwedge_{a\in F} d(x,a) \geq d(x,y)$. Hence $x \in c^d_\alpha([y]) \iff d(x,y) \geq \alpha$ and $y \in c^d_\beta([z]) \iff d(y,z) \geq \beta$ implies

$d(x,z) \geq d(x,y) * d(y,z) \geq \alpha * \beta$, i.e. $x \in c^d_{\alpha*\beta}([z])$. For left-continuity, let $x \in c^d_\alpha(\mathbb{F})$ for all $\alpha \in A \subseteq L$. Then for all $\alpha \in A$ we have $\bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y) \geq \alpha$ and therefore also $\bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y) \geq \bigvee A$, i.e. $x \in c^d_{\bigvee A}(\mathbb{F})$.

To show the pretopologicalness, let $x \in \bigcap_{j\in J} c^d_\alpha(\mathbb{F}_j)$. Then for all $j \in J$ and all $\epsilon \lhd \alpha$ there is $F^\epsilon_j \in \mathbb{F}_j$ such that for all $y_j \in F^\epsilon_j$, $d(x,y_j) \geq \epsilon$. But then $F = \bigcup_{j\in J} F^\epsilon_j \in \bigwedge_{j\in J} \mathbb{F}_j$ and for all $y \in F$ we have $d(x,y) \geq \epsilon$. Hence $\bigvee_{F\in\bigwedge_{j\in J}\mathbb{F}_j} \bigwedge_{y\in F} d(x,y) \geq \epsilon$, i.e. $x \in c^d_\epsilon(\bigwedge_{j\in J}\mathbb{F}_j)$. This is true for all $\epsilon \lhd \alpha$ and by left-continuity and using $\alpha = \bigvee\{\epsilon : \epsilon \lhd \alpha\}$ we obtain $x \in c^d_\alpha(\bigwedge_{j\in J}\mathbb{F}_j)$. □

Proposition 3.4. *Let $f : (X,d) \longrightarrow (X',d')$ be an* L*-metric morphism. Then $f : (X,\overline{c^d}) \longrightarrow (X',\overline{c^{d'}})$ is continuous.*

Proof. Let $x \in c^d_\alpha(\mathbb{F})$. Then

$$\alpha \leq \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y) \leq \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d'(f(x),f(y))$$
$$= \bigvee_{F\in\mathbb{F}} \bigwedge_{y'\in f(F)} d'(f(x),y') \leq \bigvee_{G\in f(\mathbb{F})} \bigwedge_{y'\in G} d'(f(x),y').$$

Hence $f(x) \in c^{d'}_\alpha(f(\mathbb{F}))$. □

Hence we have a functor from L-MET into L-PreMET-CTS. We note that this functor is injective on objects. If $d \neq d'$ then, without loss of generality, there are $x,y \in X$ such that $d(x,y) \not\leq d'(x,y)$. Then $x \in c^d_{d(x,y)}([y])$ but $x \notin c^{d'}_{d(x,y)}([y])$.

Proposition 3.5. *Let $(X,\overline{c}) \in$ |L-PreMET-CTS| and define*

$$d^c(x,y) = \bigvee_{x\in c_\alpha([y])} \alpha.$$

Then $(X,d^c) \in$ |L-MET|.

Proof. (LM1) follows from (LC1) and (LM2) follows from the distributivity of the quantale operation over arbitrary joins and the transitivity as

$$d^c(x,y) * d^c(y,z) = \bigvee_{x\in c_\alpha([y]),\, y\in c_\beta([z])} \alpha * \beta \leq \bigvee_{x\in c_{\alpha*\beta}([z])} \alpha * \beta \leq d^c(x,z).$$

□

Proposition 3.6. *Let $f : (X,\overline{c}) \longrightarrow (X',\overline{c'})$ be a morphism in* L-PreMET-CTS*. Then $f : (X,d^c) \longrightarrow (X',d^{c'})$ is an* L*-metric morphism.*

Proof. We have for $x,y \in X$

$$d^c(x,y) = \bigvee_{x\in c_\alpha([y])} \alpha \leq \bigvee_{f(x)\in c'_\alpha([f(y)])} \alpha = d^{c'}(f(x),f(y)).$$

□

Proposition 3.7. *Let $(X,d) \in$ |L-MET|. Then $d^{(c^d)} = d$.*

Proof. We note again that $x \in c^d_\alpha([y]) \iff d(x,y) \geq \alpha$. Hence $d^{(c^d)}(x,y) = \bigvee_{x \in c^d_\alpha([y])} \alpha = \bigvee_{d(x,y)\geq\alpha} \alpha = d(x,y)$. □

Proposition 3.8. *Let* $(X, \overline{c}) \in |\mathsf{L}\text{-}\mathsf{PreMET}\text{-}\mathsf{CTS}|$. *Then* $c^{(d^c)}_\alpha(\mathbb{F}) \subseteq c_\alpha(\mathbb{F})$.

Proof. Let $x \in c^{(d^c)}_\alpha(\mathbb{F})$ and let $\epsilon \lhd \alpha$. Using the interpolation property, there is $F_\epsilon \in \mathbb{F}$ such that $\epsilon \lhd \delta \lhd \bigwedge_{y \in F_\epsilon} d^c(x,y)$ for some $\delta \in L$. Hence for all $y \in F_\epsilon$ we have $\epsilon \lhd \bigvee_{x \in c_\beta([y])} \beta$, i.e. for all $y \in F_\epsilon$ there is $\beta \geq \epsilon$ such that $x \in c_\beta([y])$. But then also $x \in c_\epsilon([y])$. Hence we have for all $\epsilon \lhd \alpha$ a set $F_\epsilon \in \mathbb{F}$ such that $x \in \bigcap_{y \in F_\epsilon} c_\epsilon([y])$. Now from $F_\epsilon \in \mathbb{F}$ we conclude $[F_\epsilon] = \bigwedge_{y \in F_\epsilon}[y] \leq \mathbb{F}$ and from pretopologicalness we conclude that for all $\epsilon \lhd \alpha$ there is $F_\epsilon \in \mathbb{F}$ such that $x \in c_\epsilon([F_\epsilon]) \subseteq c_\epsilon(\mathbb{F})$. This is true for all $\epsilon \lhd \alpha$ and from the left-continuity we again conclude $x \in c_{\bigvee\{\epsilon \lhd \alpha\}}(\mathbb{F}) = c_\alpha(\mathbb{F})$. □

Theorem 3.9. *The category* $\mathsf{L}\text{-}\mathsf{MET}$ *can be coreflectively embedded into the category* $\mathsf{L}\text{-}\mathsf{PreMET}\text{-}\mathsf{CTS}$.

Remark 3.10. In [14] for the case $\mathsf{L} = (\Delta^+, \leq, *)$ we embedded the category of probabilistic metric spaces into the category of probabilistic convergence tower spaces in a different way. Following Tardiff [29], we define for an L-metric space (X,d), $\epsilon > 0$ and $\varphi \in \Delta^+$ the (φ,ϵ)-neighbourhood of $x \in X$ by

$$N^{\varphi,\epsilon}_x = \{y \in X \ : \ d(x,y)(u+\epsilon) + \epsilon \geq \varphi(u) \ \forall u \in [0, \frac{1}{\epsilon})\}.$$

and define the φ-neighbourhood filter of $x \in X$, $\mathbb{N}^\varphi_x$ as the filter generated by the sets $N^{\varphi,\epsilon}_x$, $\epsilon > 0$. If we define

$$x \in \tilde{c}^d_\varphi(\mathbb{F}) \iff \mathbb{F} \geq \mathbb{N}^\varphi_x$$

then we obtain a left-continuous and pretopological L-convergence tower space $(X, \overline{\tilde{c}^d})$. In order to show that this L-convergence tower space coincides with the L-convergence tower space $(X, \overline{c^d})$, we need the following results from [29]. For $\varphi \in \Delta^+$ and $0 \leq \epsilon \leq 1$ we define $\varphi^\epsilon \in \Delta^+$ by

$$\varphi^\epsilon(u) = \begin{cases} 0 & \text{if} \quad u = 0 \\ (\varphi(u+\epsilon)+\epsilon) \wedge 1 & \text{if} \quad 0 < u \leq \frac{1}{\epsilon} \\ 1 & \text{if} \quad u > \frac{1}{\epsilon}. \end{cases}$$

Clearly then $\varphi \leq \varphi^\epsilon$ and Tardiff [29] shows that $y \in N^{\varphi,\epsilon}_x$ if and only if $d(x,y)^\epsilon \geq \varphi$ and $\varphi \geq \psi$ if and only if for all $\epsilon > 0$ we have $\varphi^\epsilon \geq \psi$. The last assertion implies that for $\varphi \in \Delta^+$ we have $\varphi = \bigwedge_{\epsilon>0} \varphi^\epsilon$. We will need the following results.

Lemma A. *Let* $\varphi_j \in \Delta^+$ *for all* $j \in J$ *and let* $0 \leq \epsilon \leq 1$. *Then* $(\bigvee_{j\in J} \varphi_j)^\epsilon = \bigvee_{j \in J}(\varphi^\epsilon_j)$ *and* $(\bigwedge_{j\in J}\varphi_j)^\epsilon = \bigwedge_{j\in J}(\varphi^\epsilon_j)$.

Proof. We only show the second assertion, the first one being similar. For $u = 0$ or $u > \frac{1}{\epsilon}$ the assertion is obvious. Let $0 < u \le \frac{1}{\epsilon}$. Then we have

$$
\begin{aligned}
(\bigwedge_{j\in J}\varphi_j)^{\epsilon}(u) &= \left((\bigwedge_{j\in J}\varphi_j)(u+\epsilon)+\epsilon\right)\wedge 1\\
&= \left((\sup_{v<u+\epsilon}\inf_{j\in J}\varphi_j(v))+\epsilon\right)\wedge 1\\
&= \left(\sup_{v<u+\epsilon}(\inf_{j\in J}\varphi_j(v)+\epsilon)\right)\wedge 1\\
&= \sup_{v<u+\epsilon}\inf_{j\in J}((\varphi_j(v)+\epsilon)\wedge 1)\\
&= \sup_{w+\epsilon<u+\epsilon}\inf_{j\in J}((\varphi_j(w+\epsilon)+\epsilon)\wedge 1)\\
&= \sup_{w<u}\inf_{j\in J}\varphi_j^{\epsilon}(w)\\
&= \bigwedge_{j\in J}\varphi_j^{\epsilon}(u).
\end{aligned}
$$

□

Lemma B (cf. [20], Proposition 1.8.29). *Let $\mathbb{U} \in \mathbb{U}(X)$ be an ultrafilter and $f : X \longrightarrow L$ be a mapping. Then $\bigvee_{U\in\mathbb{U}}\bigwedge_{y\in U} f(y) = \bigwedge_{U\in\mathbb{U}}\bigvee_{y\in U} f(y)$.*

Proof. It is easy to show that for $U, V \in \mathbb{U}$ we have $\bigwedge_{y\in U} f(y) \le \bigvee_{y\in V} f(y)$ and hence $\bigvee_{U\in\mathbb{U}}\bigwedge_{y\in U} f(y) \le \bigwedge_{U\in\mathbb{U}}\bigvee_{y\in U} f(y)$. For the converse inequality, let $\alpha \lhd \bigwedge_{U\in\mathbb{U}}\bigvee_{y\in U} f(y)$. Then for all $V \in \mathbb{U}$ there is $y \in V$ such that $f(y) \ge \alpha$. With $V_\alpha = \{y \in X : f(y) \ge \alpha\}$ then $V \cap V_\alpha \ne \varnothing$ for all $V \in \mathbb{U}$ and hence $V_\alpha \in \mathbb{U}$. Therefore $\bigvee_{V\in\mathbb{U}}\bigwedge_{y\in V} f(y) \ge \bigwedge_{y\in V_\alpha} f(y) \ge \alpha$. By the complete distributivity then $\bigwedge_{U\in\mathbb{U}}\bigvee_{y\in U} f(y) \le \bigvee_{V\in\mathbb{U}}\bigwedge_{y\in V} f(y)$. □

Let now $x \in c^d_\varphi(\mathbb{F})$ and $\mathbb{U} \ge \mathbb{F}$ be an ultrafilter and $\psi \lhd \varphi$. Then, by Lemma B,

$$\bigwedge_{U\in\mathbb{U}}\bigvee_{y\in U} d(x,y) = \bigvee_{U\in\mathbb{U}}\bigwedge_{y\in U} d(x,y) \ge \varphi \rhd \psi,$$

and hence, for all $U \in \mathbb{U}$ there is $y^\psi \in U$ such that $d(x,y^\psi)^\epsilon \ge d(x,y^\psi) \ge \psi$. But this means $y^\psi \in N_x^{\psi,\epsilon}$ for all $\epsilon > 0$. So we conclude that for all $U \in \mathbb{U}$ we have $N_x^{\psi,\epsilon} \cap U \ne \varnothing$, and hence, $\mathbb{U}$ being an ultrafilter, $N_x^{\psi,\epsilon} \in \mathbb{U}$ for all $\epsilon > 0$, i.e. $\mathbb{N}_x^\psi \le \mathbb{U}$. Therefore $x \in \tilde{c}^d_\psi(\mathbb{U})$ for all $\psi \lhd \varphi$ and from the left-continuity then also $x \in \tilde{c}^d_\varphi(\mathbb{U})$. This is true for all ultrafilters $\mathbb{U} \ge \mathbb{F}$ and hence, by pretopologicalness, $x \in \tilde{c}^d_\varphi(\mathbb{F})$.

Conversely, let $\mathbb{F} \ge \mathbb{N}_x^\varphi$. Then, for $\epsilon > 0$, we have

$$\bigvee_{F\in\mathbb{F}}\bigwedge_{y\in F} d(x,y)^\epsilon \ge \bigwedge_{y\in N_x^{\varphi,\epsilon}} d(x,y)^\epsilon \ge \varphi.$$

From Lemma A we conclude

$$\begin{aligned}\varphi &\leq \bigwedge_{\epsilon>0} \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y)^{\epsilon} \\ &= \bigwedge_{\epsilon>0} (\bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y))^{\epsilon} \\ &= \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y)\end{aligned}$$

and we have $x \in c^d_{\varphi}(\mathbb{F})$.

4. The isomorphy of L-MET and L-MET-CTS

We introduce the following axiom for an L-convergence tower space. We say that $(X,\overline{c}) \in |\mathsf{L\text{-}CTS}|$ satisfies the axiom (LM) if for all $\mathbb{U} \in \mathbb{U}(X)$ and all $\alpha \in L$ we have

(LM) $\quad x \in c_\alpha(\mathbb{U}) \iff \forall U \in \mathbb{U}, \beta \lhd \alpha \exists y \in U \text{ s.t. } x \in c_\beta([y]).$

This axiom was introduced in [7] for probabilistic convergence spaces in the sense of Richardson and Kent [26].

Theorem 4.1. *Let $(X,d) \in |\mathsf{L\text{-}MET}|$. Then $(X,\overline{c^d})$ satisfies (LM).*

Proof. Let $\mathbb{U} \in \mathbb{U}(X)$ and let $\alpha \in L$. Let first $x \in c^d_\alpha(\mathbb{U})$ and let $U \in \mathbb{U}$ and $\beta \lhd \alpha$. Then there is $F_\beta \in \mathbb{U}$ such that for all $y \in F_\beta$ we have $d(x,y) \geq \beta$. Choose $y \in U \cap F_\beta$. Then

$$\bigvee_{F\in[y]} \bigwedge_{z\in F} d(x,z) \geq \bigwedge_{z\in U\cap F_\beta} d(x,z) \geq \beta,$$

i.e. $x \in c_\beta([y])$.

Conversely, let for all $U \in \mathbb{U}$, $\beta \lhd \alpha$ there is $y = y_\beta \in U$ such that $x \in c^d_\beta([y])$, i.e. such that $\bigvee_{F\in[y]} \bigwedge_{z\in F} d(x,z) \geq \beta$. Let further $F \in \mathbb{U}^x = \bigwedge_{x\in c^d_\beta(\mathbb{F})} \mathbb{F}$. Then, for $U \in \mathbb{U}$ in particular $F \in [y]$, i.e. $y \in F \cap U$. Hence $\mathbb{U} \vee \mathbb{U}^x$ exists and because $\mathbb{U}$ is an ultrafilter, we get $\mathbb{U} \geq \bigwedge_{x\in c^d_\beta(\mathbb{F})} \mathbb{F}$. As $(X,\overline{c^d})$ is pretopological, we conclude $c^d_\beta(\mathbb{U}) \supseteq \bigcap_{x\in c^d_\beta(\mathbb{F})} c^d_\beta(\mathbb{F})$ and we have $x \in c^d_\beta(\mathbb{U})$. This is true for any $\beta \lhd \alpha$ and by left-continuity we obtain $x \in c^d_\alpha(\mathbb{U})$. □

Proposition 4.2. *Let $(X,\overline{c}) \in |\mathsf{L\text{-}PreMET\text{-}CTS}|$ satisfy the axiom (LM). Then $c^{(d^c)}_\alpha(\mathbb{F}) = c_\alpha(\mathbb{F})$.*

Proof. Let $\mathbb{U} \in \mathbb{U}(X)$ be an ultrafilter and let $x \in c_\alpha(\mathbb{U})$. By the axiom (LM) we obtain, for $\beta \lhd \alpha$ that $N^x_\beta = \{y \in X : x \in c_\beta([y])\}$ satisfies $N^x_\beta \cap U \neq \varnothing$ for all $U \in \mathbb{U}$ and hence $N^x_\beta \in \mathbb{U}$. Furthermore, for $x \in c_\beta([y])$ we have $d^c(x,y) \geq \beta$. Hence

$$\bigvee_{U\in\mathbb{U}} \bigwedge_{y\in U} d^c(x,y) \geq \bigwedge_{y\in N^x_\beta} d^c(x,y) = \bigwedge_{x\in c_\beta([y])} d^c(x,y) \geq \beta.$$

This is true for all $\beta \lhd \alpha$ and hence also $\bigwedge_{y \in N_\beta^x} d^c(x,y) = \bigwedge_{x \in c_\beta([y])} d^c(x,y) \geq \alpha$, which is equivalent to $x \in c_\alpha^{(d^c)}(\mathbb{U})$. Hence we have shown $c_\alpha(\mathbb{U}) \subseteq c_\alpha^{(d^c)}(\mathbb{U})$ for all $\mathbb{U} \in \mathbb{U}(X)$ and because both $(X, \overline{c})$ and $(X, \overline{c^{(d^c)}})$ are pretopological, we have for $\mathbb{F} \in \mathbb{F}(X)$ that $c_\alpha(\mathbb{F}) \subseteq c_\alpha^{(d^c)}(\mathbb{F})$. The converse implication is always true and so we have equality. □

If we denote the subcategory of L-PreMET-CTS with objects the L-metric spaces that satisfy the axiom (LM) by L-MET-CTS, then we obtain the following main result.

Theorem 4.3. *The categories* L-MET-CTS *and* L-MET *are isomorphic.*

Remark 4.4. For $\mathsf{L} = (\Delta^+, \leq, *)$ with a continuous triangle function $*$, we introduced in [14] for $(X,d) \in |\mathsf{L}\text{-CTS}|$ a different axiom (PM): For all $\mathbb{U} \in \mathbb{U}(X)$, all $\varphi \in \Delta^+$ and all $x \in X$ we have

$$x \in c_\varphi(\mathbb{U}) \iff \forall U \in \mathbb{U}, \epsilon > 0 \ \exists y \in U \text{ s.t.} \bigvee_{x \in c_\psi([y])} \psi(u+\epsilon)+\epsilon \geq \varphi(u) \forall u \in [0, \frac{1}{\epsilon}).$$

With the notation of this paper and of Remark 3.10 then $d^{(\tilde{c}^d)} = d$ and if $(X, \overline{c})$ is $*$-transitive, left-continuous and pretopological, then $\tilde{c}_\varphi^{(d^c)}(\mathbb{F}) = c_\varphi(\mathbb{F})$ for all $\varphi \in \Delta^+$ and all $\mathbb{F} \in \mathbb{F}(X)$. It follows from this, that for an L-convergence tower space $(X, \overline{c})$ that is $*$-transitive, left-continuous and pretopological, the axioms (PM) and (LM) are equivalent. In fact, if (PM) is true, then $\tilde{c}_\varphi^{(d^c)} = c_\varphi$ and hence, using Remark 3.10, then also $c_\varphi^{(d^c)} = c_\varphi$ and as d^c is an L-metric on X we know that $(X, \overline{c}) = (X, \overline{c^{(d^c)}})$ satisfies (LM). A similar argument shows that (LM) implies (PM).

5. L-partial metric spaces as L-convergence tower spaces

An L-*partial metric space* [17, 22] is a pair (X,p) of a set X and a mapping $p : X \times X \longrightarrow L$ with

(LPM1) $p(x,y) \leq p(x,x)$ for all $x, y \in X$;
(LPM2) $p(x,y) = p(y,x)$ for all $x, y \in X$;
(LPM3) $p(x,y) * (p(y,y) \to p(y,z)) \leq p(x,z)$.

Morphisms are defined as in L-MET and the category of L-partial metric spaces is denoted by L-PMET.

For $\mathsf{L} = ([0,\infty], \geq, +)$, an L-partial metric space is a partial metric space [22]. These spaces were introduced motivated by problems in computer science, where the self-distances $d(x,x)$ are not always zero, [21]. Independently and in much wider generality, L-partial metric spaces were introduced and studied under the name M-valued sets in [11, 12]. For $\mathsf{L} = (\Delta^+, \leq, *)$, L-partial metric spaces are called *probabilistic partial metric spaces* in [31] and *fuzzy partial metric spaces* in [30].

We note that $p(y,z) \leq p(y,y) \to p(y,z)$ and hence (LPM3) implies the transitivity axiom (LM2).

In the sequel, we need to adapt the definition of L-convergence tower spaces. We relax the axiom (LC1) and replace it by

(wLC1) $x \in c_\alpha([x])$ whenever $c_\alpha([x]) \neq \varnothing$.

An L-*partial convergence tower space* is a pair $(X, \overline{c})$ which satisfies the axioms (wLC1), (LC2), (LC3) and (LC4). With morphisms as defined before, we denote the category of L-partial convergence tower spaces by L-PCTS.

We will use the same functors as defined above to embed the category of L-partial metric spaces into the category of L-partial convergence tower spaces. Only few adaptations are necessary, so that we simply repeat the results and only prove the modifications.

Proposition 5.1. *Let* $(X, p) \in |\mathsf{L}\text{-}\mathsf{PMET}|$. *Define*

$$x \in c^p_\alpha(\mathbb{F}) \iff \bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} p(x, y) \geq \alpha.$$

Then $(X, \overline{c^p}) \in |\mathsf{L}\text{-}\mathsf{PCTS}|$.

Proof. We only need to prove (wLC1). As before, we can show that $y \in c^p_\alpha([x])$ if and only if $p(x, y) \geq \alpha$. If $y \in c^p_\alpha([x])$, then $p(x, y) \geq \alpha$ and then by (LPM1) we have $p(x, x) \geq \alpha$, i.e. $x \in c^p_\alpha([x])$. □

We call an L-(partial) convergence tower space $(X, \overline{c})$ *strongly $*$-transitive* if

(LST) $x \in c_{\alpha*(\mathbb{E}(y)\to\beta)}([z])$ whenever $x \in c_\alpha([y])$ and $y \in c_\beta([z])$,

where $\mathbb{E}(y) = \bigvee_{y \in c_\gamma([y])} \gamma$. It is called *symmetric* if

(LS) $x \in c_\alpha([y])$ whenever $y \in c_\alpha([x])$.

Lemma 5.2. *If the* L*-(partial) convergence tower space* $(X, \overline{c})$ *is strongly $*$-transitive, then it is transitive.*

Proof. This follows from $\alpha * \beta \leq \alpha * (\mathbb{E}(y) \to \beta)$ and the axiom (LC3). □

Proposition 5.3. *Let* $(X, p) \in |\mathsf{L}\text{-}\mathsf{PMET}|$. *Then* $(X, \overline{c^p})$ *is strongly $*$-transitive, left-continuous, symmetric and pretopological.*

Proof. We need to check the strong $*$-transitivity (LST) and the symmetry (LS). For (LST) we first note that $\mathbb{E}(y) = \bigvee_{y \in c^p_\beta([y])} \beta = \bigvee_{\beta \leq p(y,y)} \beta = p(y, y)$. Let $x \in c^p_\alpha([y])$ and $y \in c^p_\beta([z])$. Then $\alpha \leq p(x, y)$ and $\beta \leq p(y, z)$ and hence $\alpha * (\mathbb{E}(y) \to \beta) \leq p(x, y) * (p(y, y) \to p(y, z)) \leq p(x, z)$ and hence $x \in p_{\alpha*(\mathbb{E}(y)\to\beta)}([z])$. For (LS), let $x \in c^p_\alpha([y])$. Then $p(x, y) = p(y, x) \geq \alpha$ and hence $y \in c^p_\alpha([x])$. □

Proposition 5.4. *Let* $f : (X, d) \longrightarrow (X', d')$ *be an* L-PMET*-morphism. Then* $f : (X, \overline{c^d}) \longrightarrow (X', \overline{c^{d'}})$ *is continuous.*

Hence we have a functor from L-PMET into the category of strongly $*$-transitive, left-continuous, symmetric and pretopological L-partial convergence tower spaces, L-$\mathsf{PrePMET}$-PCTS. Again, this functor is injective on objects.

In the sequel, we have to restrict the lattice context quite strongly. We say that the quantale $\mathsf{L} = (L, \le, *)$ satisfies the axiom (DM2) if for all non-empty index sets J we have
(DM2) $\quad \alpha \to \bigvee_{j\in J} \beta_j = \bigvee_{j\in J}(\alpha \to \beta_j)$ for all $\alpha, \beta_j \in L (j \in J)$.

Examples for quantales that satisfy (DM2) are complete MV-algebras and also $\mathsf{L} = ([0,\infty], \ge, +)$. In general, $\mathsf{L} = (\Delta^+, \le, *)$ does not satisfy (DM2). We show this with the following example.

Example 5.5. We consider the triangle function induced by the product t-norm defined by $\varphi * \psi(u) = \varphi(u) \cdot \psi(u)$ for all $u \in [0,\infty]$. We define for each natural number $n \in \mathbb{N}$ the distance distribution function $\varphi_n \in \Delta^+$ by $\varphi_n(u) = n(u-1)$ for $1 \le u \le 1+\frac{1}{n}$. Then $\bigvee_{n\in\mathbb{N}} \varphi_n = \varepsilon_1$ and hence $\varepsilon_1 \to \bigvee_{n\in\mathbb{N}} \varphi_n = \varepsilon_0$. On the other hand it is not difficult to show that $\varepsilon_1 \to \varphi_n = \varphi_n$ and hence $\bigvee_{n\in\mathbb{N}}(\varepsilon_1 \to \varphi_n) = \varepsilon_1$.

Proposition 5.6. *Let the quantale* $\mathsf{L} = (L, \le, *)$ *satisfy the axiom (DM2). Let* $(X, \overline{c}) \in |\mathsf{L\text{-}PrePMET\text{-}PCTS}|$ *and define*

$$p^c(x,y) = \bigvee_{x\in c_\alpha([y])} \alpha.$$

Then $(X, p^c) \in |\mathsf{L\text{-}PMET}|$.

Proof. (LPM1) We have, using (wLC1),

$$p^c(x,y) = \bigvee_{y\in c_\alpha([x])} \alpha = \left\{ \begin{array}{ll} \perp & \text{if } c_\alpha([x]) = \varnothing \\ \le \bigvee_{x\in c_\alpha([x])} \alpha & \text{if } c_\alpha([x]) \ne \varnothing \end{array} \right\} \le p^c(x,x).$$

(LPM2) follows from the symmetry (LS). We show (LPM3). First we note that $\mathbb{E}(y) = \bigvee_{y\in c_\beta([y])} \beta = p^c(y,y)$. Let now $x \in c_\alpha([y])$ and $y \in c_\beta([z])$. With the axiom (LST) then $x \in c_{\alpha*(\mathbb{E}(y)\to\beta)}([z])$ and hence $\alpha * (\mathbb{E}(y) \to \beta) \le p^c(x,z)$. We conclude with (DM2) and by the distributivity of the quantale operation over joins

$$\bigvee_{x\in c_\alpha([y])} \bigvee_{y\in c_\beta([z])} (\alpha*(\mathbb{E}(y) \to \beta)) = (\bigvee_{x\in c_\alpha([y])} \alpha)* \left(\mathbb{E}(y) \to \bigvee_{y\in c_\beta([z])} \beta \right) \le p^c(x,z),$$

which is nothing else than $p^c(x,y) * (p^c(y,y) \to p^c(y,z)) \le p^c(x,z)$. $\square$

Proposition 5.7. *Let the quantale* L *satisfy the axiom (DM2). Let* $f : (X, \overline{c}) \longrightarrow (X', \overline{c'})$ *be continuous. Then* $f : (X, p^c) \longrightarrow (X', p^{c'})$ *is an* $\mathsf{L\text{-}PMET}$*-morphism.*

Proposition 5.8. *Let the quantale* L *satisfy the axiom (DM2). Let* $(X,p) \in |\mathsf{L\text{-}PMET}|$. *Then* $p^{(c^p)} = p$.

Proposition 5.9. *Let the quantale* L *satisfy the axiom (DM2). Let* $(X, \overline{c}) \in |\mathsf{L\text{-}PMET\text{-}PCTS}|$. *Then* $c_\alpha^{(p^c)}(\mathbb{F}) \subseteq c_\alpha(\mathbb{F})$.

Theorem 5.10. *Let the quantale* L *satisfy the axiom (DM2). Then the category* $\mathsf{L\text{-}PMET}$ *can be coreflectively embedded into the category* $\mathsf{L\text{-}PrePMET\text{-}PCTS}$.

We extend the axiom (LM) to L-partial convergence tower spaces.

(LM) $\forall \mathbb{U} \in \mathbb{U}(X), \alpha \in L$ we have
$x \in c_\alpha(\mathbb{U}) \iff \forall U \in \mathbb{U}, \beta \lhd \alpha \exists y \in U$ s.t. $x \in c_\beta([y])$.

The proofs of the following results do not make use of the axiom (LC1) and hence they carry over to L-partial metric spaces without any alterations.

Theorem 5.11. *Let $(X,p) \in |\mathsf{L\text{-}PMET}|$. Then $(X, \overline{c^p})$ satisfies (LM).*

Proposition 5.12. *Let the quantale L satisfy the axiom (DM2) and let $(X,\overline{c}) \in |\mathsf{L\text{-}PrePMET\text{-}PCTS}|$ satisfy the axiom (LM). Then $c_\alpha^{(p^c)}(\mathbb{F}) = c_\alpha(\mathbb{F})$.*

If we denote the subcategory of $\mathsf{L\text{-}PrePMET\text{-}PCTS}$ with objects the L-partial metric spaces that satisfy the axiom (LM) by $\mathsf{L\text{-}PMET\text{-}PCTS}$, then we obtain the following main result.

Theorem 5.13. *Let the quantale L satisfy the axiom (DM2). Then the categories $\mathsf{L\text{-}PMET\text{-}PCTS}$ and $\mathsf{L\text{-}PMET}$ are isomorphic.*

6. L-metrization of L-convergence tower groups

Let $(X,\cdot)$ be a group with neutral element e. For filters $\mathbb{F},\mathbb{G} \in \mathbb{F}(X)$, the filter $\mathbb{F} \odot \mathbb{G}$ is generated by the sets $F \odot G = \{xy \ : \ x \in F, y \in G\}$ for $F \in \mathbb{F}$ and $G \in \mathbb{G}$ and the filter $\mathbb{F}^{-1}$ is generated by the sets $F^{-1} = \{x^{-1} \ : \ x \in F\}$ for $F \in \mathbb{F}$.

Definition 6.1 (see [4])**.** A triple $(X,\cdot,\overline{c})$, where $(X,\cdot)$ is a group and $(X,\overline{c})$ is an L-convergence tower space, is called an L-*convergence tower group* if for all $x,y \in X$ and all $\mathbb{F},\mathbb{G} \in \mathbb{F}(X)$

(LCTGM) $xy \in c_{\alpha*\beta}(\mathbb{F} \odot \mathbb{G})$ whenever $x \in c_\alpha(\mathbb{F})$ and $y \in c_\beta(\mathbb{G})$;
(LCTGI) $x^{-1} \in c_\alpha(\mathbb{F}^{-1})$ whenever $x \in c_\alpha(\mathbb{F})$.

A mapping $f : X \longrightarrow X'$, where $(X,\cdot,\overline{c})$ and $(X',\cdot',\overline{c'})$ are L-convergence tower groups, is called an $\mathsf{L\text{-}CTG}$-morphism, if f is a homomorphism between the groups $(X,\cdot),(X',\cdot')$ and a morphism in $\mathsf{L\text{-}CTS}$. The category of L-convergence tower groups and $\mathsf{L\text{-}CTG}$-morphisms is denoted by $\mathsf{L\text{-}CTG}$.

For $L = \{0,1\}$, we obtain classical convergence groups [18, 6], for $\mathsf{L} = ([0,1],\leq,*)$ we obtain the probabilistic convergence groups in the sense of [16] and for $\mathsf{L} = (\Delta^+,\leq,*)$ we obtain the probabilistic convergence groups of [4]. For $\mathsf{L} = ([0,\infty],\geq,+)$ we obtain limit tower groups [3]. An L-convergence tower group is a stratified $\{0,1\}\{0,1\}L$-convergence tower group in the definition of [5].

Lemma 6.2. *Let $(X,\cdot,\overline{c}) \in |\mathsf{L\text{-}CTG}|$ and let $\alpha \in L$, $x \in X$ and $\mathbb{F} \in \mathbb{F}(X)$. Then $x \in c_\alpha(\mathbb{F})$ if and only if $e \in c_\alpha([x^{-1}] \odot \mathbb{F})$.*

Proof. If $x \in c_\alpha(\mathbb{F})$ then by (LC1) and (LCTGM) we conclude $e = x^{-1}x \in c_{\top*\alpha}([x^{-1}] \odot \mathbb{F}) = c_\alpha([x^{-1}] \odot \mathbb{F})$. Conversely, if $e \in c_\alpha([x^{-1}] \odot \mathbb{F})$, then $x = xe \in c_{\top*\alpha}([x] \odot [x^{-1}] \odot \mathbb{F}) = c_\alpha(\mathbb{F})$. □

Lemma 6.3. *Let $(X,\cdot,\overline{c}) \in |\mathsf{L\text{-}CTG}|$. Then $(X,\overline{c})$ is $*$-transitive.*

Proof. Let $x \in c_\alpha([y])$ and $y \in c_\beta([z])$. Then $e \in c_\alpha([x^{-1}] \odot [y])$ and $e \in c_\beta([y^{-1}] \odot [z])$. By (LCTGM) then $e = ee \in c_{\alpha*\beta}([x^{-1}] \odot [y] \odot [y^{-1}] \odot [z]) = c_{\alpha*\beta}([x^{-1}] \odot [z])$ and hence $x \in c_{\alpha*\beta}([z])$. □

Definition 6.4. A triple $(X, \cdot, d)$ is called an L-*metric group* if d is *invariant*, i.e. if $d(x,y) = d(xz,yz) = d(zx,zy)$ for all $x,y,z \in X$. A group homomorphism $f : (X,\cdot) \longrightarrow (X', \cdot')$ between the L-metric groups $(X,\cdot,d), (X',\cdot',d')$ is called an L-METG-morphism if it is an L-metric morphism between (X,d) and (X',d'). The category of L-metric groups is denoted by L-METG.

This definition is motivated by the following result, where we use, for an L-metric $d : X \times X \longrightarrow L$ on X, the *product* L-*metric* on $X \times X$ defined by $d \circledast d : (X \times X) \times (X \times X) \longrightarrow L$, $d \circledast d((x,y),(x',y')) = d(x,x') * d(y,y')$.

Lemma 6.5. *Let $(X,\cdot)$ be a group and let $d : X \times X \longrightarrow L$ be an L-metric which is symmetric, i.e. for which $d(x,y) = d(y,x)$ for all $x,y \in X$ holds. Then the L-metric d is invariant if and only if the mappings $m : X \times X \longrightarrow X$, $m(x,y) = xy$ and $i : X \longrightarrow X$, $i(x) = x^{-1}$ are L-metric morphisms.*

Proof. Let first d be an invariant metric on X. Then using the transitivity, we obtain $d \circledast d((x,y),(x',y')) = d(x,x') * d(y,y') = d(xy,x'y) * d(x'y,x'y') \le d(xy,x'y') = d(m(x,y),m(x',y'))$, i.e. multiplication is an L-metric morphism. Furthermore, using the symmetry of d, we obtain $d(x,y) = d(y^{-1}xx^{-1}, y^{-1}yx^{-1})$ $= d(y^{-1},x^{-1}) = d(x^{-1},y^{-1})$, i.e. inversion is an L-metric morphism.

For the converse, we note that, multiplication being an L-metric morphism, we have for all $x,x',y,y' \in X$, $d(x,y) * d(x',y') = d \circledast d((x,x'),(y,y')) \le d(xy,x'y')$. In particular, we have $d(x,y) = d(x,y) * d(z,z) \le d(xz,yz)$ and similarly $d(xz,yz) = d(xz,yz) * d(z^{-1},z^{-1}) \le d(xzz^{-1},yzz^{-1}) = d(x,y)$. Similarly we can show that $d(x,y) = d(zx,zy)$ and hence d is invariant. □

We call an L-convergence tower group $(X,\overline{c})$ L-*metrizable* if there is a symmetric and invariant L-metric d on X such that $\overline{c} = \overline{c^d}$.

Theorem 6.6. *An L-convergence tower group $(X,\cdot,\overline{c})$ is L-metrizable if and only if it is left-continuous, pretopological, symmetric and satisfies the axiom (LM).*

Proof. We have seen above that if there is an L-metric d such that $\overline{c} = \overline{c^d}$, then $(X,\overline{c})$ is left-continuous, pretopological and satisfies the axiom (LM). Symmetry of $(X,\overline{c^d})$ follows easily from the symmetry of d. Conversely, let $(X,\overline{c})$ be left-continuous, pretopological, symmetric and satisfy the axiom (LM). Then $d^c(x,y) = \bigvee_{x \in c_\alpha([y])} \alpha$ is a symmetric L-metric on X that satisfies $\overline{c^{d^c}} = \overline{c}$. We only need to show that d^c is invariant. To this end, we note that by (LCTGM) and (LC1) we have for $x,y,z \in X$ that $x \in c_\alpha([y])$ if and only if $xz \in c_\alpha([yz])$. Hence $d^c(xz,yz) = \bigvee_{xz \in c_\alpha([yz])} \alpha = \bigvee_{x \in c_\alpha([y])} \alpha = d^c(x,y)$. Similarly, we see that $d^c(zx,zy) = d^c(x,y)$ and hence d^c is invariant. □

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