Generalized Lindley Power Series family of Distributions

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ABSTRACT

In this paper, we introduce a new class of distributions by compounding the generalized class of Lindley and power series family of distributions. This new class of distributions contains several lifetime subclasses, such as Lindley Poisson, Lindley geometric, Lindley logarithmic and Lindley binomial classes of distributions. It also can generate as many statistical distributions such as power Lindley Poisson, power Lindley geometric, power Lindley logarithmic and power Lindley binomial distributions. The proposed class has flexibility in the sense that it can generates many new lifetime distributions as well as some existing distributions. For the proposed class, several properties, such as survival functions, hazard rate functions, limiting behavior, quantile functions, moments, and distribution of order statistics are derived. The method of maximum likelihood estimation will be used to estimate the model parameters of this new class. We will study the Lindley logarithmic distribution in some details including simulation as an example of the proposed class. At the end, we will demonstrate applications of three real data sets to show the flexibility and potential of the new class of distributions.

- **Keywords:** Generalized Lindley power series distributions, Lindley power series distributions, power Lindley logarithmic distribution

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1. Introduction

Consider the lifetime of a system with N components where the life of each component is a positive continuous random variable, say X_i. Then, the life of such a system can be modeled as a non-negative random variable $X = \min\{X_i\}_{i=1}^{N}$ or $Y = \max\{X_i\}_{i=1}^{N}$ based on whether the components are in a series or parallel. In some applications, we have an unknown random number of causes for the failure of a system and each will live randomly according to the distribution X_i. The discrete random variable N can have several distributions, such as zero-truncated Poisson, geometric, logarithmic, binomial, and the generalized power series of distributions. The continuous random variables $X_i, i = 1, \ldots, N$ are independent and can be have any lifetime distribution, such as exponential, gamma, Weibull, or Lindley. We should emphasize that we have a choice of two distribution types: Weibull type and Lindley type. The relation between these two types was established in a larger class called, "a class of Lindley and Weibull distributions", as discussed in Alkarni [1]. However, the Weibull types of distributions cannot exhibit bathtub shapes for their hazard rate functions. Lindley [2] suggested an alternative for exponential distribution and developed in many types of Lindley distributions. These distributions have been able to overcome the weakness of the Weibull distribution by exhibiting all types of hazard rate functions.

In this paper, we introduce the generalized Lindley power series of distributions since the Weibull power series of distributions were studied intensively by Alkarni [3]. The proposed class of distributions is obtained by compounding the generalized class of Lindley with the power series of distributions. We call it generalized Lindley power series (GLPS) family of distributions. The GLPS class of distributions is a flexible family and contains many types of Lindley compounded with discrete distributions (truncated at one), such as Lindley Poisson (Gui et al. [4]), Lindley geometric (Zakerzadeh and Mahmoudi [5], and Lindley logarithmic distribution (Liyanage and Pararai [6]). In a similar manner, some classes of distributions are proposed in the literature: Lindley power series (LPS) class of distributions (Liyanage and Pararai [6]), generalized extended Weibull power series family of distributions (Alkarni [3]), exponentiated extended Weibull-power series class of distributions (Tahmasebi and Jafari [7]), the exponential-power series of distributions (Chahkandi and Ganjali [8]), Weibull-power series of distributions (Morais and Barreto-Souza [9]), generalized exponential-power series of distributions (Mahmoudi and Jafari [10]), complementary exponential power series (Flores et al. [11]), extended Weibull-power series of distributions (Silva et al. [12]), double bounded Kumaraswamy power series (Bidram and Nekoukhou [13]), Burr XII power series (Silva and Cordeiro [14]), generalized linear failure rate-power series of distributions (Alamatsaz and Shams [15]), Birnbaum Saunders power series of distribution (Bourguignon et al. [16]), linear failure rate-power series (Mahmoudi and Jafari [17]), and complementary extended Weibull-power series (Cordeiro and Silva [18]). To compound continuous distribution with discrete distribution, Nadarajah et al. [19] introduced the package Compounding in R software (R Development Core Team, [20]).

We introduce the GLPS class of distributions for the following reasons:

(1) Lindley distributions are widely used in modeling lifetime data.
(2) The GLPS class of distributions exhibits some interesting behavior with non-monotonic failure rates, such as bathtub, upside bathtub, and increasing-decreasing-increasing failure rates.
(3) The theoretical results of this paper can be used to obtain new useful distributions with all mathematical properties verified in this study.

The proposed family of distributions can be applied to other fields, such as business, environment, actuarial science, biomedical studies, demography, and industrial reliability. This family contains several subclasses and lifetime models as special cases. In addition, it gives us the flexibility of choosing any compound lifetimes for modeling any type of lifetime data.

The remainder of this paper is organized as follows. In Section 2, we define the generalized class of Lindley distributions, and demonstrate that many existing models can be deduced as special cases of the proposed unified model. In Section 3, we define the GLPS class of distributions in terms of cumulative distribution functions (cdf) and introduce some special cases of some existing classes. In Section 4, we provide the general properties of the GLPS class, including probability density function (pdf), survival and hazard rate function (hrf), quantile function, moments, moments generating functions, and distribution of order statistics. The estimation of the GLPS parameters is investigated in Section 5 using the method of maximum likelihood estimation and a large sample inference. In Section 6, special subclasses and some special distributions are introduced along with the flexible mathematical forms of their properties. In Section 7, the Lindley logarithmic distribution is introduced with some mathematical properties as a special application of GLPS to show the benefits of this class. In Section 8, we present some real data to illustrate the applicability and flexibility of the GLPS distributions. Finally, some concluding remarks are addressed in Section 9.

2. The generalized Lindley class of distributions

Lindley distribution is among the most widely used lifetime distributions in terms of reliability. Many modifications have been suggested for Lindley distribution to improve the shape of the hazard rate function. Peng and Yan [21] presented many references on this matter. The following definition introduces the generalized Lindley (GL) class of distributions, which generates most of the existing Lindley types of distributions and can be used to generate new distributions.

Definition. Let $\mathrm{H}(x;\xi)$ be a non-negative function that depends on a non-negative parameter vector $\xi > 0$, the GL class of distributions is defined by its cumulative cdf, G, as follows,

$$G(x;\beta,\delta,\xi) = 1 - \left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)};\beta,\delta,\xi,x>0. \tag{2.1}$$

The corresponding probability distribution function (pdf) becomes,

$$g(x;\beta,\delta,\xi) = \frac{\beta^2}{\beta+\delta}(1+\delta\,\mathrm{H}(x;\xi))\,\mathrm{h}(\mathrm{x};\xi)\,e^{-\beta\mathrm{H}(\mathrm{x};\xi)};\beta,\delta,\xi,x>0, \tag{2.2}$$

where $\mathrm{h}(\mathrm{x};\xi)$ is the first derivative of $\mathrm{H}(x;\xi)$.

Many Lindley distributions can be written in form (2.1) depending on the parameters β,δ and the choice of the function $\mathrm{H}(x;\xi)$, see Alkarni [1]. In the following sub section, we present some special distributions of this class.

2.1 Special cases

2.1.1 Lindley distribution

If $H(x)=x$ and $\delta=1$. Then the cdf in (2.1) is then,

$$F(x;\beta) = 1 - \left(\frac{\beta+1+\beta x}{\beta+1}\right)e^{-\beta x}, \beta,x>0, \tag{2.1.1}$$

which is the cdf of Lindley distribution introduced by Lindley [2]. From (2.2), the pdf is given by

$$f(x;\beta)=\frac{\beta^2}{\beta+1}(1+x)e^{-\beta x},\beta,x>0. \quad (2.1.2)$$

2.1.2 Two parameters Lindley distribution

If $H(x)=x$. Then the cdf in (2.1) is then,

$$F(x;\beta,\delta)=1-\left(\frac{\beta+\delta+\beta\delta x}{\beta+\delta}\right)e^{-\beta x},\beta,\delta,x>0, \quad (2.1.3)$$

which is the cdf of Lindley distribution introduced by Shanker et al. [22]. From (2.2), the pdf is given by

$$f(x;\beta,\delta)=\frac{\beta^2}{\beta+\delta}(1+\delta x)e^{-\beta x},\beta,\delta,x>0. \quad (2.1.4)$$

2.1.3 Power Lindley distribution

If $H(x)=x^{\alpha},\delta=1$. Then the cdf in (2.1) is then,

$$F(x;\beta,\alpha)=1-\left(\frac{\beta+1+\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}},\beta,\alpha,x>0, \quad (2.1.5)$$

which is the cdf of power Lindley distribution introduced by Ghitany et al. [23]. From (2.2), the pdf is given by

$$f(x;\beta,\alpha)=\frac{\alpha\beta^2}{\beta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}},\beta,\alpha,x>0. \quad (2.1.6)$$

2.1.4 Extended power Lindley distribution

If $H(x)=x^{\alpha}$. Then the cdf in (2.1) is then,

$$F(x;\beta,\delta,\alpha)=1-\left(\frac{\beta+\delta+\beta\delta x^{\alpha}}{\beta+\delta}\right)e^{-\beta x^{\alpha}},\beta,\delta,\alpha,x>0, \quad (2.1.7)$$

which is the cdf of extended power Lindley distribution introduced by Alkarni [24]. From (2.2), the pdf is given by

$$f(x;\beta,\delta,\alpha)=\frac{\alpha\beta^2}{\beta+\delta}(1+\delta x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}},\beta,\delta,\alpha,x>0. \quad (2.1.8)$$

3. The GLPS family

In this section, we derive the family of GLPS distributions by compounding the GL class of distributions and the power series distributions.

Let N be a zero-truncated discrete random variable having a power series distributions with the following probability mass function:

$$p_n = p(N = n) = \frac{a_n \theta^n}{c(\theta)}, \quad n = 1, 2, \ldots,$$

where $a_n \geq 0$ depends only on n, $c(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$, and $\theta \in (0, s)$ is chosen in such a way that $c(\theta)$ is finite. The power series family of distributions, including Poisson, geometric, logarithmic, and binomial distribution, are presented in Johnson et al. [25]. Useful quantities such as $a_n, c(\theta), c'(\theta)$ and $c''(\theta)$ are introduced in Table 1.

Table 1: Useful quantities for some power series distributions

Distribution	a_n	$c(\theta)$	$c'(\theta)$	$c''(\theta)$	Parameter Space
Poisson	$\frac{1}{n}$	e^{θ}	e^{θ}	e^{θ}	$(0, \infty)$
Geometric	1	$\frac{\theta}{1-\theta}$	$\frac{1}{(1-\theta)^2}$	$\frac{2}{(1-\theta)^3}$	$(0, 1)$
Logarithmic	$\frac{1}{n}$	$-\log(1-\theta)$	$\frac{1}{(1-\theta)^2}$	$\frac{2}{(1-\theta)^3}$	$(0, 1)$
Binomial	$\binom{m}{n}$	$(1+\theta)^m - 1$	$m(1+\theta)^{m-1}$	$m(m-1)(1+\theta)^{m-2}$	N^+

Given N, let $X_{(1)} = \min(X_1, \ldots, X_N)$, where $X_i, i = 1, \ldots, N$ are independent and identically distributed (iid) random variables following (2.1). Then, the cdf of $X_{(1)} | N = n$ is given by

$$F_{X_{(1)}|N=n}(\mathrm{x}) = 1 - \left(\frac{\beta + \delta + \beta\delta \mathrm{H}(x;\xi)}{\beta + \delta} \right)^n e^{-n\beta \mathrm{H}(x;\xi)}, \quad \mathrm{x} > 0, \mathrm{n} \geq 1.$$

The GLPS distribution is then defined by the marginal cdf of $X_{(1)}$ which is given by

$$F_{GLPS}(x) = F_{X_{(1)}}(x) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{c(\theta)} \left[1 - \left(\frac{\beta + \delta + \beta\delta \mathrm{H}(x;\xi)}{\beta + \delta} \right)^n e^{-n\beta \mathrm{H}(x;\xi)} \right]$$

$$= 1 - \frac{c\left[\theta \left(\frac{\beta + \delta + \beta\delta \mathrm{H}(x;\xi)}{\beta + \delta} \right) e^{-\beta \mathrm{H}(x;\xi)} \right]}{c(\theta)}, \quad \theta, \beta, \delta, \xi, x > 0, \qquad (3.1)$$

which can be written as

$$F_{GLPS}(x) = \sum_{n=1}^{\infty} p_n (1 - (G(x))^n) = 1 - \frac{c(\theta(1 - G(x)))}{c(\theta)}, \quad \mathrm{x} > 0. \qquad (3.2)$$

Remark. Let $X_{(n)} = \max\{X_i\}_{i=1}^{N}$, then the cdf of $X_{(n)}$ is given by,

$$F_{X_{(n)}}(x) = \sum_{n=1}^{\infty} p_n (G(x))^n = \frac{c(\theta G(x))}{c(\theta)}, \tag{3.3}$$

$$= \frac{c\left[\theta\left(1-\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right)\right]}{c(\theta)}, \theta,\beta,\delta,\xi,x>0. \tag{3.4}$$

Note that if $\mathrm{H}(x) = \mathrm{x}, \delta = 1$, then (3.1) is reduced to

$F_{LPS}(x) = 1 - \frac{c\left(\theta\left(\frac{\beta+1+\beta x}{\beta+1}e^{-\beta x}\right)\right)}{c(\theta)}$, the Lindley power series (LPS) class of distributions introduced by Liyanage and Pararai [6].

Based on the choice of $a_n, c(\theta), H(\mathrm{x})$ and δ with form (3.1) and (3.4), this class covers the entire compound truncated discrete distributions with all the Lindley types of distributions for a series and parallel components.

4. General properties

4.1 Density, survival, and hazard rate functions

The pdfs associated with (3.1), (3.2), (3.3) and (3.4), respectively, are given by

$$f_{GPLS}(x) = \theta\,\mathrm{g(x)}\frac{c'(\theta(1-G(x)))}{c(\theta)}, \tag{4.1.1}$$

$$=\theta\frac{\beta^2}{\beta+\delta}(1+\delta\,\mathrm{H}(x;\xi))\,\mathrm{h}(\mathrm{x};\xi)\,e^{-\beta\mathrm{H}(\mathrm{x};\xi)}\frac{c'\left[\theta\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right]}{c(\theta)} \tag{4.1.2}$$

and

$$f_{X_{(n)}}(x) = \theta\,\mathrm{g(x)}\frac{c'(\theta\,\mathrm{G(x)})}{c(\theta)}, \tag{4.1.3}$$

$$=\theta\frac{\beta^2}{\beta+\delta}(1+\delta\,\mathrm{H}(x;\xi))\,\mathrm{h}(\mathrm{x};\xi)\,e^{-\beta\mathrm{H}(\mathrm{x};\xi)}c'\left[\theta\left(1-\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right)\right].$$

(4.1.4)

The survival functions (sf) are given by

$$s_{GPLS}(x)=\frac{c(\theta(1-G(x))}{c(\theta)}=\frac{c\left[\theta\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right]}{c(\theta)},\qquad(4.1.5)$$

and

$$s_{X_{(n)}}(x)=1-\frac{c(\theta G(x))}{c(\theta)}=1-\frac{c\left[\theta\left(1-\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right)\right]}{c(\theta)}.\qquad(4.1.6)$$

The corresponding hazard rate functions (hrf) are

$$\tau_{GPLS}(x)=\frac{f_{X_{(1)}}(\mathrm{x})}{s_{X_{(1)}}(x)}=\theta\,\mathrm{g}(\mathrm{x})\frac{c'(\theta(1-G(x)))}{c(\theta(1-G(x)))},\qquad(4.1.7)$$

$$=\theta\frac{\beta^2}{\beta+\delta}(1+\delta\,\mathrm{H}(x;\xi))\,\mathrm{h}(\mathrm{x};\xi)\,e^{-\beta\mathrm{H}(\mathrm{x};\xi)}\frac{c'\left[\theta\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right]}{c\left[\theta\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right]},\qquad(4.1.8)$$

and

$$\tau_{X_{(n)}}(x)=\frac{f_{X_{(n)}}(\mathrm{x})}{s_{X_{(n)}}(x)}=\theta g(x)\frac{c'(\theta G(x))}{c(\theta)-c(\theta G(x))},\qquad(4.1.9)$$

$$=\theta\frac{\beta^2}{\beta+\delta}(1+\delta\,\mathrm{H}(x;\xi))\,\mathrm{h}(\mathrm{x};\xi)\,e^{-\beta\mathrm{H}(\mathrm{x};\xi)}\frac{c'\left[\theta\left(1-\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right)\right]}{c(\theta)-c\left[\theta\left(1-\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right)\right]}.$$

(4.1.10)

The limiting distribution of the GLPS when $\theta\to 0^+$ is

$$\lim_{\theta\to 0^+} F_{GPLS}(x) = \lim_{\theta\to 0^+} 1-\frac{c(\theta(1-G(x)))}{c(\theta)} = 1-\lim_{\theta\to 0^+}\frac{c(\theta(1-G(x)))}{c(\theta)}$$

$$=1-\lim_{\theta\to 0^+}\frac{\sum_{n=1}^{\infty} a_n\theta^n(1-G(x))^n}{\sum_{n=1}^{\infty} a_n\theta^n},$$ using the L'Hôpital's rule, we obtain

$$=1-\lim_{\theta\to 0^+}\frac{a_1(1-G(x))+\sum_{n=2}^{\infty} na_n\theta^{n-1}(1-G(x))^n}{a_1+\sum_{n=2}^{\infty} na_n\theta^{n-1}}$$

$$=1-(1-G(x))=G(x)=1-\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}.$$

The pdf of GLPS distributions can be expressed as an infinite number of linear combinations of densities of the order statistics. Given that $c'(\theta)=\sum_{n=1}^{\infty} na_n\theta^{n-1}$, therefore,

$$f_{GLPS}(x)=\theta\,\mathrm{g}(\mathrm{x})\frac{c'(\theta(1-G(\mathrm{x})))}{c(\theta)}=\sum_{n=1}^{\infty} p(\mathrm{N}=\mathrm{n})\,\mathrm{g}_{Y_{(1)}}(x;n),$$

where $\mathrm{g}_{Y_{(1)}}(x;n)$ is the pdf of $Y_{(1)}=\min(Y_1,\ldots,Y_n)$, given by

$$g_{Y_{(1)}}(x;n)=ng(x)(1-G(x))^{n-1}$$

$$=n\frac{\beta^2}{\delta+\beta}(1+\delta\,\mathrm{H}(x;\xi))\,\mathrm{h}(\mathrm{x};\xi)\,e^{-\beta\mathrm{H}(\mathrm{x};\xi)}\left[\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right]^{n-1},$$ hence

$$f_{GLPS}(x)=\sum_{n=1}^{\infty}\frac{na_n\theta^n}{c(\theta)}\frac{\beta^2}{(\delta+\beta)^n}(1+\delta\,\mathrm{H}(x;\xi))\,\mathrm{h}(\mathrm{x};\xi)\left[\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)\right]^{n-1}e^{-n\beta\mathrm{H}(\mathrm{x};\xi)}$$

(4.1.1)

4. 2 Moments, and moments generating function

The r^{th} moment of a random variable X from the GLPS distribution, μ_r' is given by

$$\mu_r'=\sum_{n=1}^{\infty} P[N=n]\int_0^{\infty} x^r g_{Y_{(1)}}(x)dx=\sum_{n=1}^{\infty}\frac{an\theta^n}{c(\theta)}\int_0^{\infty} x^r g_{Y_{(1)}}(x)dx,$$

which can be obtained for any function $H(x)$.

The moment generating functions (mgf) are obtained as follows:

$$M_{GLPS}(t)=\sum_{n=1}^{\infty}\frac{an\theta^n}{c(\theta)}\int_0^{\infty}e^{tx}g_{Y_{(1)}}(x)dx.$$

Using the series expansion $e^{tx}=\sum_{k=0}^{\infty}\frac{t^k x^k}{k!}$, the above expressions are reduced to

$$M_{GLPS}(t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{t^k}{k!}\frac{an\theta^n}{c(\theta)}\mu_k^{'}.$$

4. 3 Quantile function and order statistics

In this section, the quantile function of GLPS distributions will be derived. Let X be a random variable with cdf as in (3.1). The quantile function, i.e., $Q_{X_{(1)}}(p)$, is the root of the equation $F_{X_{(1)}}(Q_{X_{(1)}}(\mathrm{p}))=\mathrm{p},\mathrm{p}\in(0,1)$. Therefore,

$$\frac{c\left[\theta\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(\mathrm{Q}_{X_{(1)}}(\mathrm{p}))}{\beta+\delta}\right)e^{-\beta\mathrm{H}(\mathrm{Q}_{X_{(1)}}(\mathrm{p}))}\right]}{c(\theta)}=1-p,$$

multiply both sides by $c(\theta)$ and apply $c^{-1}(.)$ for both sides, then multiply both sides by $\frac{-\theta e^{-\delta-\beta}}{\beta+\delta}$ leads to

$$-\beta-\delta-\beta\delta\,\mathrm{H}(\mathrm{Q}_{X_{(1)}}(\mathrm{p}))e^{-\beta-\delta-\beta\delta\mathrm{H}(\mathrm{Q}_{X_{(1)}}(\mathrm{p}))}=-\frac{(\beta+\delta)c^{-1}((1-p)c(\theta))}{\theta e^{\beta+\delta}}$$

Let $z(p)=-\delta-\beta-\delta\beta\,\mathrm{H}(\mathrm{Q}_{X_{(1)}}(\mathrm{p}))$, then we have

$z(p)e^{z(p)}=-\frac{(\beta+\delta)c^{-1}((1-p)c(\theta))}{\theta e^{\beta+\delta}}$. Then the solution for $z(p)$ is

$$z(p)=W\left[-\frac{(\beta+\delta)c^{-1}((1-p)c(\theta))}{\theta e^{\beta+\delta}}\right]$$

Where $W(.)$ is the negative branch of Lambert W functions. See Corless et al. [26]. Hence, the quantile functions of GLPS is given by

$$Q_{GLPS}(p)=Q_{X_{(1)}}(p)=H^{-1}\left[-\frac{1}{\delta}-\frac{1}{\beta}-\frac{1}{\delta\beta}W\left(-\frac{(\beta+\delta)c^{-1}((1-p)c(\theta))}{\theta e^{\beta+\delta}}\right)\right]. \qquad (4.2.1)$$

Order statistics are among the most fundamental tools in non-parametric statistics and inference. These can be used to tackle estimation problems and hypothesis tests in many ways. The pdf of the k^{th} order statistics from a random sample $X_1,...,X_n$ from the GLPS is given by

$$f_{k:n}(x)=\frac{n!}{(k-1)!(n-k)!}f_{GLPS}(x)[F_{GLPS}(x)]^{k-1}[1-F_{GLPS}(x)]^{n-k},$$

$$=\frac{n!}{(k-1)!(n-k)!}f_{GLPS}(x)\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i[F_{GLPS}(x)]^{k+i-1}. \qquad (4.2.2)$$

The associate cdf can be obtained as

$$F_{k:n}(x)=\frac{n!}{(k-1)!(k-i)!}\sum_{i=0}^{n-k}\frac{\binom{n-k}{i}(-1)^i}{k+i}[F_{GLPS}(x)]^{k+i}. \qquad (4.2.3)$$

5. Estimation and inference

Let $X_1,...,X_n$ be a random sample, with the observed value $x=(x_1,...,x_n)$ obtained from the GLPS distribution with parameters θ,β,δ and $\boldsymbol{\xi}$. Let $\Theta=(\theta,\beta,\delta,\boldsymbol{\xi})^T$ be the $p\times 1$ parameter vector. The log likelihood function is given by

$$l_n=l_n(\Theta,x)=n\log\theta+2n\log\beta-n\log(\beta+\delta)-n\log(\mathrm{c}(\theta))+\sum_{i=1}^{n}\log(1+\delta H(x_i))$$

$$+\sum_{i=1}^{n}\log(\mathrm{h}(x_i))-\beta\sum_{i=1}^{n}(\mathrm{H}(x_i))+\sum_{i=1}^{n}\log\left\{c'\left[\theta\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}\right]\right\}. \qquad (5.1)$$

Consider $p_i=\left(\frac{\beta+\delta+\beta\delta\,\mathrm{H}(x;\xi)}{\beta+\delta}\right)e^{-\beta\mathrm{H}(x;\xi)}$, $p'_{i\beta}=\partial p_i/\partial\beta, p'_{i\delta}=\partial p_i/\partial\delta$. Then the score function is given by $U_n(\Theta)=(\partial l_n/\partial\theta,\partial l_n/\partial\beta,\partial l_n/\partial\delta,\partial l_n/\partial\boldsymbol{\xi})^T$.

$$\frac{\partial l_n}{\partial\theta}=\frac{n}{\theta}-\frac{nc'(\theta)}{c(\theta)}+\sum_{i=1}^{n}\frac{p_i c''(\theta p_i)}{c'(\theta p_i)},$$

$$\frac{\partial l_n}{\partial\beta}=\frac{2n}{\beta}-\frac{n}{\beta+\delta}-\sum_{i=1}^{n}H(x_i)+\theta\sum_{i=1}^{n}\frac{c''(\theta p_i)p'_{i\beta}}{c'(\theta p_i)},$$

$$\frac{\partial l_n}{\partial\delta}=-\frac{n}{\beta+\delta}+\sum_{i=1}^{n}\frac{H(x_i)}{1+\delta H(x_i)}-\theta\sum_{i=1}^{n}\frac{c''(\theta p_i)p'_{i\delta}}{c'(\theta p_i)},$$

$$\frac{\partial l_n}{\partial\boldsymbol{\xi}_k}=\sum_{i=1}^{n}\frac{\delta}{1+\delta H(x_i)}\frac{\partial\mathrm{H}(x_i)}{\partial\boldsymbol{\xi}_k}+\sum_{i=1}^{n}\frac{1}{h(x_i)}\frac{\partial h(x_i)}{\partial\boldsymbol{\xi}_k}-\beta\sum_{i=1}^{n}\frac{\partial H(x_i)}{\partial\boldsymbol{\xi}_k}+\theta\sum_{i=1}^{n}\frac{c''(\theta p_i)}{c'(\theta p_i)}\frac{\partial p_i}{\partial\boldsymbol{\xi}_k},$$

Where ξ_k is the kth element of the vector ξ.

The maximum likelihood estimation (MLE) of Θ say Θ is obtained by solving the nonlinear system $U_n(\mathrm{x};\Theta)=0$. Since this nonlinear system of equations does not have a closed form solution, any numerical method such as the Newton-Raphson procedure can be used. For the interval estimation and hypothesis tests on the model parameters, we require the following observed information matrix:

$$I_n(\Theta)=-\begin{bmatrix} I_{\theta\theta} & I_{\theta\beta} & I_{\theta\delta} & | & I_{\theta\xi}^T \\ I_{\beta\theta} & I_{\beta\beta} & I_{\beta\delta} & | & I_{\beta\xi}^T \\ I_{\delta\theta} & I_{\delta\beta} & I_{\delta\delta} & | & I_{\delta\xi}^T \\ — & — & — & — & — \\ I_{\theta\xi} & I_{\beta\xi} & I_{\delta\xi} & | & I_{\xi\xi} \end{bmatrix},$$

where the elements of $I_n(\Theta)$ are the second partial derivatives of $U_n(\Theta)$. Under the standard regular conditions for the large sample approximation in Cox and Hinkley [27], which was fulfilled for the proposed model, the distribution of Θ is approximately $N_p(\Theta, J_n(\Theta)^{-1})$, where $J_n(\Theta)=\mathrm{E}[\mathrm{I}_n(\Theta)]$. Whenever the parameters are in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\Theta-\Theta)$ is $N_p(0, J(\Theta)^{-1})$, where $J(\Theta)^{-1}=\lim_{n\to\infty} n^{-1} I_n(\Theta)$ is the unit information matrix and p is the number of parameters of the distribution. The asymptotic multivariate normal $N_p(\Theta, I_n(\Theta)^{-1})$ distribution of Θ can be used to approximate the confidence interval for the parameters, the hazard rate, and the survival functions. An $100(1-\gamma)$ asymptotic confidence interval for parameter Θ_i is given by

$$(\Theta_i - Z_{\frac{\gamma}{2}}\sqrt{I^{ii}}, \Theta_i + Z_{\frac{\gamma}{2}}\sqrt{I^{ii}}),$$

where I^{ii} is the (i,i) diagonal element of $I_n(\Theta)^{-1}$ for $i=1,\dots,p$ and $Z_{\frac{\gamma}{2}}$ is the quantile $1-\frac{\gamma}{2}$ of the standard normal distribution.

6. Special subclasses

In this section, we present some subclasses of GLPS distributions. Using (3.2), (3.3), (4.1.1), (4.1.3), (4.1.5), (4.1.6), (4.1.7), and (4.1.9), we provide the forms of the cdf, pdf, sf, and hrf for $X_{(1)}$ and $X_{(\mathrm{n})}$ for Poisson, geometric, logarithmic, and binomial distributions compounded with any continuous lifetime distributions defined in (2.1).

6.1 The Lindley Poisson class of distributions

The Lindley Poisson (LP) class of distributions is a special case of the GLPS class, with $a_n = \frac{1}{n!}$ and $c(\theta) = e^{\theta} - 1, \theta > 0.$ Table 2 shows the cdf, pdf, sf, and hrf for this class in both (series and parallel) systems.

Table 2: cdf, pdf, sf, and hrf for the LP class

$X_{(1)} = \min(X_1,...,X_N)$	$X_{(n)} = \max(X_1,...,X_N)$
$F(x) = \frac{e^{\theta} - e^{\theta(1-G(x))})}{e^{\theta} - 1}$ $f(x) = \frac{\theta g(x) e^{\theta(1-G(x))}}{e^{\theta} - 1}$ $s(x) = \frac{e^{\theta(1-G(x))} - 1}{e^{\theta} - 1}$ $\tau(x) = \frac{\theta g(x) e^{\theta(1-G(x))}}{e^{\theta(1-G(x))} - 1}$	$F(x) = \frac{e^{-\theta(1-G(x))} - e^{-\theta}}{1 - e^{-\theta}}$ $f(x) = \frac{\theta g(x) e^{-\theta(1-G(x))}}{1 - e^{-\theta}}$ $s(x) = \frac{1 - e^{e^{-\theta(1-G(x))}}}{1 - e^{-\theta}}$ $\tau(x) = \frac{\theta g(x) e^{e^{-\theta(1-G(x))}}}{1 - e^{e^{-\theta(1-G(x))}}}$

The Lindley Poisson distribution, which was proposed by Gui et al. [4], is a special case of the LP class with $G(x)$ and $g(x)$ the cdf and pdf of Lindley distribution as in (2.1.1) and (2.1.2), respectively.

6.2 The Lindley geometric class of distributions

The Lindley geometric (LG) class of distributions is a special case of GLPS class, with $a_n = 1$ and $c(\theta) = \frac{\theta}{1-\theta}, \theta \in (0,1).$ Table 3 shows the cdf, pdf, sf, and hrf for this class in both (series and parallel) systems.

Table 3: cdf, pdf, sf, and hrf for the LG class

$X_{(1)} = \min(X_1,...,X_N)$	
$F(x) = \frac{G(x)}{1-\theta(1-G(x))}$ $f(x) = \frac{(1-\theta)g(x)}{(1-\theta(1-G(x)))^2}$ $s(x) = \frac{(1-\theta)(1-G(x))}{1-\theta(1-G(x))}$ $\tau(x) = \frac{g(x)}{[1-\theta(1-G(x))][1-G(x)]}$	$F(x) = \frac{(1-\theta)G(x)}{1-\theta G(x)}$ $f(x) = \frac{(1-\theta)g(x)}{(1-\theta G(x))^2}$ $s(x) = 1 - \frac{(1-\theta)G(x)}{1-\theta G(x)}$ $\tau(x) = \frac{(1-\theta)g(x)}{(1-\theta G(x))(1-G(x))}$

The Lindley geometric distribution was introduced and studied by Zakerzadeh and Mahmoudi [5] belongs to the LG with $G(x)$ and $g(x)$ the cdf and pdf of Lindley distribution as in (2.1.1) and (2.1.2), respectively.

6.3 The Lindley logarithmic class of distributions

The Lindley logarithmic (LL) class of distributions is a special case of GLPS class, with $a_n = \frac{1}{n}$ and $c(\theta) = -\log(1-\theta), \theta \in (0,1)$. Table 4 shows the cdf, pdf, sf, and hrf for this class in both (series and parallel) systems.

Table 4: cdf, pdf, sf, and hrf for the LL class

$X_{(1)} = \min(X_1, ..., X_N)$	$X_{(n)} = \max(X_1, ..., X_N)$
$F(x) = 1 - \frac{\log(1-\theta(1-\mathrm{G(x)}))}{\log(1-\theta)}$ $f(x) = \frac{-\theta g(x)}{(1-\theta(1-\mathrm{G(x)}))\log(1-\theta)}$ $s(x) = \frac{\log(1-\theta(1-\mathrm{G(x)}))}{\log(1-\theta)}$ $\tau(x) = \frac{-\theta g(x)}{\log(1-\theta(1-\mathrm{G(x)}))[1-\theta(1-\mathrm{G(x)})]}$	$F(x) = \frac{\log(1-\theta\mathrm{G(x)})}{\log(1-\theta)}$ $f(x) = \frac{-\theta g(x)}{(1-\theta\mathrm{G(x)})\log(1-\theta)}$ $s(x) = 1 - \frac{\log(1-\theta\mathrm{G(x)}))}{\log(1-\theta)}$ $\tau(x) = \frac{\theta g(x)}{(1-\theta\mathrm{G(x)})[\log(1-\theta\mathrm{G(x)}) - \log(1-\theta)]}$

The Lindley logarithmic distribution was proposed by Liyanage and Pararai [6] belongs to the LL class with $G(x)$ and $g(x)$ the cdf and pdf of Lindley distribution as in (2.1.1) and (2.1.2), respectively.

6.4 The Lindley binomial class of distributions

The Lindley binomial (LB) class of distributions is a special case of GLPS class, with $a_n = \binom{m}{n}$ and $c(\theta) = (\theta+1)^m - 1, \theta \in [0,1]$. Table 5 shows the cdf, pdf, sf, and hrf for this class in both (series and parallel) systems.

Table 5: cdf, pdf, sf, and hrf for the LB class

$X_{(1)} = \min(X_1, ..., X_N)$	$X_{(n)} = \max(X_1, ..., X_N)$
$F(x) = 1 - \frac{[\theta(1-\mathrm{G(x)})+1]^m - 1}{(\theta+1)^m - 1}$ $f(x) = \frac{\mathrm{m}\theta\,\mathrm{g(x)}(\theta(1-\mathrm{G(x)})+1)^{m-1}}{(\theta+1)^m - 1}$ $s(x) = \frac{(\theta(1-\mathrm{G(x)})+1)^m - 1}{(\theta+1)^m - 1}$ $\tau(x) = \frac{\mathrm{m}\theta\,\mathrm{g(x)}(\theta(1-\mathrm{G(x)})+1)^{m-1}}{(\theta(1-\mathrm{G(x)})+1)^m - 1}$	$F(x) = \frac{(\theta\mathrm{G(x)}+1)^m - 1}{(\theta+1)^m - 1}$ $f(x) = \frac{\mathrm{m}\theta\,\mathrm{g(x)}(\theta\mathrm{G(x)}+1)^{m-1}}{(\theta+1)^m - 1}$ $s(x) = 1 - \frac{(\theta\mathrm{G(x)}+1)^m - 1}{(\theta+1)^m - 1}$ $\tau(x) = \frac{\mathrm{m}\theta\,\mathrm{g(x)}(\theta\mathrm{G(x)}+1)^{m-1}}{(\theta+1)^m - (\theta\mathrm{G(x)}+1)^m}$

7. Power Lindley logarithmic distribution

In this section, the power Lindley logarithmic (PLL) distribution is introduced as a special case of the GLPS family of distributions. The PLL will be introduced in some details.

7.1 The model

The PLL distribution is defined directly in Table 4 with $G(x)$ and $g(x)$ to be the cdf and pdf of power Lindley distributions as in (2.1.5) and (2.1.6), respectively, Gaitany et al. [23]. In the case of GLPS $H(x) = x^{\alpha}$ and $\delta = 1$ with $a_n = \frac{1}{n}$ and $c(\theta) = -\log(1-\theta), \theta \in (0,1).$ The cdf, pdf, and hrf are given respectively by

$$F_{PLL}(x;\theta,\beta,\alpha) = 1 - \frac{\log\left(1-\theta\left(\frac{\beta+1+\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right)}{\log(1-\theta)},$$

$$f_{PLL}(x;\theta,\beta,\alpha) = \frac{\theta\alpha\beta^2}{(\beta+1)\log(1-\theta)} \frac{(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}}{\left[\theta\left(\frac{\beta+1+\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}} - 1\right]}, \text{ and}$$

$$\tau_{PLL}(x;\theta,\beta,\alpha) = \frac{\theta\alpha\beta^2(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}}{(\beta+1)\log\left[1-\theta\left(\frac{\beta+1+\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]\left[\theta\left(\frac{\beta+1+\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}} - 1\right]}.$$

Figure 1 shows the pdf and hrf of the PLL distribution for the selected parameter values.

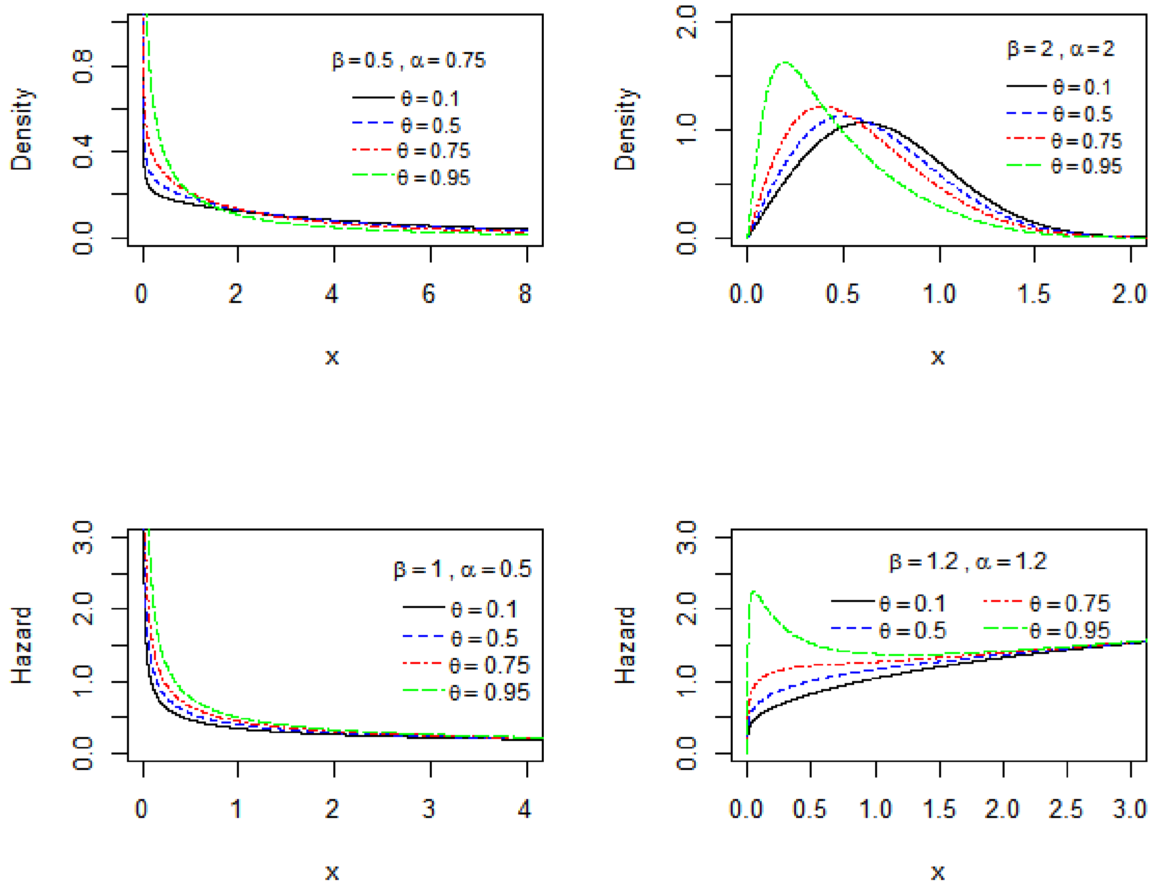

Figure 1. Plots of the density and hazard rate functions of the PLL for different values of θ, β and α.

It can be seen that for $\alpha = 1$, PLL distribution is reduced to Lindley logarithmic (LL) distribution as in Liyanage and Pararai [6].

7. 2 Quantile function and order statistics

The quantile function of the PLL can be obtained by substituting in (4.2.1) with $\delta = 1, c^{-1}(\theta) = 1 - e^{-\theta}$ and $H^{-1}(x) = x^{\frac{1}{\alpha}}$ to be

$$Q_{PLL}(p) = \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W\left(-\frac{(\beta+1)(1-e^{(1-p)\log(1-\theta)})}{\theta e^{\beta+1}} \right) \right]^{\frac{1}{\alpha}}, \ 0 < \theta < 1,$$

Where $W(.)$ is the negative branch of the Lambert functions.

The pdf and cdf of the kth order statistics of the PLL distribution can be obtained directly from (4.2.2) and (4.2.3) as

$$f_{k:n}(x)=\frac{n!}{(k-1)!(k-i)!}f_{PLL}(x)\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i[F_{PLL}(x)]^{k+i-1}, \qquad (7.2.1)$$

$$F_{k:n}(x)=\frac{n!}{(k-1)!(k-i)!}f_{PLL}(x)\sum_{i=0}^{n-k}\frac{\binom{n-k}{i}(-1)^i}{k+i}[F_{PLL}(x)]^{k+i}. \qquad (7.2.2)$$

7.3 Moments and related measures

In this section, moments, moments generating function (mgf) and some related measures, such as means, variance, skewness, and kurtosis, are discussed. The moments and the mgf are presented in the following theorem.

Theorem 1. Let X be a random variable that follows the PLL distribution, then the r^{th} row moment (about the origin) is given by

$$\mu_r'=E(X^r)=-\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\sum_{j=0}^{i+1}\frac{\theta^n}{\log(1-\theta)}\binom{n-i}{i}\binom{i+1}{j}\frac{\Gamma(r/\alpha+j+1)}{(\beta+1)^n n^{r/\alpha+j+1}\beta^{r/\alpha+j-i-1}}, \qquad (7.3.1)$$

and the mgf is given by

$$M_X(t)=-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{i=0}^{n-1}\sum_{j=0}^{i+1}\frac{t^k}{k!}\frac{\theta^n}{\log(1-\theta)}\binom{n-i}{i}\binom{i+1}{j}\frac{\Gamma(k/\alpha+j+1)}{(\beta+1)^n n^{k/\alpha+j+1}\beta^{k/\alpha+j-i-1}}, \qquad (7.3.2)$$

where $\Gamma a=\int_0^{\infty}x^{a-1}e^{-x}dx$.

Proof: $\mu_r'=E(X^r)=\int_{-\infty}^{\infty}x^r f_X(x)dx$.

For $H(x)=x^{\alpha},\alpha>0$ and a power series $p[N=n]=\frac{a_n\theta^n}{c(\theta)},n=1,2,3,...,$, we have

$$\mu_r'=\sum_{n=1}^{\infty}\frac{an\theta^n}{c(\theta)}\int_0^{\infty}x^r g_{X(1)}(x)dx$$
$$=\sum_{n=1}^{\infty}\frac{n^{-1}\theta^n}{-\log(1-\theta)}\frac{n\alpha\beta^2}{(\beta+1)^n}\int_0^{\infty}x^{r+\alpha-1}(1+x^{\alpha})(1+\beta(1+x^{\alpha}))^{n-1}e^{-n\beta x^{\alpha}}dx,$$

using $(1+z)^a=\sum_{i=0}^{\infty}\binom{a}{i}z^i$ and using integral by substitute, we have μ_r' proved.

The mgf of a continuous random variable X, when it exists, is given by

$$M_X(t)=\int_{-\infty}^{\infty}e^{tx}f(x)dx.$$

Using the series expansion, $e^{tx} = \sum_{k=0}^{\infty} \frac{t^k x^k}{k!}$, we get the mgf using the same ideas above.

Therefore, the mean and the variance of the PLL distributions, respectively, are

$$\mu = \mu_1' = -\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\sum_{j=0}^{i+1} \frac{\theta^n}{\log(1-\theta)} \binom{n-i}{i} \binom{i+1}{j} \frac{\Gamma(1/\alpha + j + 1)}{(\beta+1)^n n^{1/\alpha+j+1} \beta^{1/\alpha+j-i-1}},$$

and

$$\sigma^2 = \mu_2' - \mu^2.$$

The *skewness* and *kurtosis* measures can be obtained from the expressions,

$$skewness = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{\sigma^3}$$

$$curtosis = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{\sigma^4},$$

upon substituting for the row moments in (7.3.1).

7.4 Maximum likelihood estimation

Let $x_{1,} x_2, ..., x_n$ be a random sample with size n obtained from the PLL distribution with parameters θ, β and α. Let $\Theta = (\theta, \beta, \alpha)^T$ be the 3×1 unknown parameter vector. From equation (5.1), the log likelihood function of the PLL distribution is given by

$$l_n = l_n(\Theta, x) = n\log\theta + n\log\alpha + 2n\log\beta - n\log(\beta+1) - n\log[\log(1-\theta)]$$
$$+\sum_{i=1}^{n}\log(1+x_i^{\alpha}) + (\alpha-1)\sum_{i=1}^{n}\log x_i - \beta\sum_{i=1}^{n}x_i^{\alpha} - \sum_{i=1}^{n}\log\left[\theta\left(\frac{1+\beta+\beta x_i^{\alpha}}{\beta+1}\right)e^{-\beta x_i^{\alpha}} - 1\right].$$

The associate score function is $U_n(\Theta) = (\partial l_n / \partial\theta, \partial l_n / \partial\beta, \partial l_n / \partial\alpha)^T$ where the elements of $U_n(\Theta)$ are given by

$$\frac{\partial l_n}{\partial \theta}=\frac{n}{\theta}+\frac{n}{(1-\theta)\log(1-\theta)}+\sum_{i=1}^{n}\frac{(1+\beta+\beta x_i^{\alpha})}{\theta(1+\beta+\beta x_i^{\alpha})-(\beta+1)e^{\beta x_i^{\alpha}}},$$

$$\frac{\partial l_n}{\partial \beta}=\frac{2n}{\beta}-\frac{n}{\beta+1}-\sum_{i=1}^{n}x_i^{\alpha}+\frac{\theta\beta}{\beta+1}\sum_{i=1}^{n}\frac{x_i^{\alpha}(2+\beta+x_i^{\alpha}+\beta x_i^{\alpha})}{[\theta(1+\beta+\beta x_i^{\alpha})-(\beta+1)e^{\beta x_i^{\alpha}}},$$

$$\frac{\partial l_n}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{1+x_i^{\alpha}}+\sum_{i=1}^{n}\log x_i-\beta\sum_{i=1}^{n}x_i^{\alpha}\log x_i$$

$$+\theta\beta^2\sum_{i}^{n}\frac{x_i^{\alpha}l\ \mathrm{og}\, x_i(1+x_i^{\alpha})}{\theta(1+\beta+\beta x_i^{\alpha})-(\beta+1)e^{\beta x_i^{\alpha}}},$$

respectively.

The maximum likelihood estimates of Θ can be obtained as a solution of $U_n(\Theta)=0$ by any numerical method, such as Newton-Raphson in R. Fisher information matrix is a 3×3 matrix consisting of the second partial derivatives of $U_n(\Theta)$ and given by

$$I_n(\Theta)=\begin{bmatrix} I_{\theta\theta} & I_{\theta\beta} & I_{\theta\alpha} \\ I_{\beta\theta} & I_{\beta\beta} & I_{\beta\alpha} \\ I_{\alpha\theta} & I_{\alpha\beta} & I_{\alpha\alpha} \end{bmatrix},$$

Where $I_{ij}(\Theta)=-E_{\theta}\left[\frac{\partial^2 \ln(x;\Theta)}{\partial\theta_i\partial\theta_j}\right]$. These elements can be obtained from R or MATLAB to get a confidence interval for the estimates.

7.5. Simulation study

In this section, the performances of the mle's estimators are discussed using their Average bias (AB), Root mean squared error (RMSE), Coverage probability of 95% confidence intervals of the parameters (CP) and Average width (AW) of 95% confidence intervals of the parameters.

Table 6 shows the comparative behavior of AB, RMSE, CP and AW. We generated 5000 random samples of different sizes for two sets of parameters using the following Lemma.

Lemma 7.5.1. Let U be a standard uniform variable between zero and one. Then the random variable

$$X=\left[-1-\frac{1}{\beta}-\frac{1}{\beta}W\left(-\frac{(\beta+1)(1-e^{(1-U)\log(1-\theta)})}{\theta e^{\beta+1}}\right)\right]^{\frac{1}{\alpha}},$$

is said to be come from the PLL distribution with parameters θ,β and α.

For each samples of size $n = 800, 1000, 2000, 3000, 4000$ and 5000 combined with two sets of parameters: $(\theta = 0.75, \beta = 0.5, \alpha = 0.95)$ and $(\theta = 0.75, \beta = 2, \alpha = 4)$, for simulation on the basis of 5000samples generated by using Lemma 7.5.1. It can be seen that as the sample size increase, the RMSE and the bias decrease toward zero. Moreover, the average confidence width decreased as the sample size increases and the coverage probabilities of the confidence interval are quite close to the nominal 95% level. We conclude that the mle's estimate and their asymptotic results can be used in inference applications such as hypothesis and confidence intervals.

Table 6: The AB, RMSE, CP and AW for varying n, θ, β and α.

		$\theta = 0.75, \beta = 0.5, \alpha = 0.95$				$\theta = 0.75, \beta = 2, \alpha = 4$			
Par.	n	AB	RMSE	CP	AW	AB	RMSE	CP	AW
θ	800	-0.134	0.315	0.983	1.508	-0.066	0.205	0.975	0.937
	1000	-0.085	0.222	0.997	1.100	-0.058	0.191	0.961	0.819
	2000	-0.054	0.176	0.958	0.709	-0.028	0.136	0.925	0.523
	3000	-0.033	0.142	0.935	0.545	-0.015	0.106	0.923	0.408
	4000	-0.024	0.122	0.935	0.459	-0.011	0.091	0.929	0.348
	5000	-0.018	0.105	0.938	0.402	-0.011	0.082	0.938	0.311
β	800	0.0308	0.112	0.998	0.567	0.038	0.234	0.997	1.148
	1000	0.0240	0.097	0.999	0.520	0.035	0.221	0.994	1.026
	2000	0.0133	0.082	0.990	0.363	0.009	0.174	0.963	0.717
	3000	0.0062	0.071	0.975	0.295	0.000	0.145	0.957	0.584
	4000	0.0038	0.063	0.961	0.254	0.000	0.127	0.953	0.505
	5000	0.0023	0.056	0.955	0.227	0.000	0.114	0.952	0.452
α	800	-0.008	0.063	0.974	0.325	-0.008	0.276	0.986	1.391
	1000	-0.004	0.029	0.989	0.160	-0.006	0.260	0.980	1.246
	2000	-0.001	0.025	0.966	0.114	0.009	0.211	0.970	0.887
	3000	0.000	0.022	0.965	0.094	0.016	0.183	0.959	0.727
	4000	0.000	0.020	0.958	0.081	0.012	0.161	0.951	0.628
	5000	0.000	0.018	0.958	0.073	0.006	0.143	0.951	0.561

8. Applications

In this section, we fit the $PLL(\theta,\beta,\alpha)$ distribution to three real data sets and compare it with some of the other distributions, such as the Lindley logarithmic distribution (LL) introduced by Liyanage and Pararai [6], the Weibull logarithmic distribution (LW) introduced by Ciumara and Preda [28], exponential logarithmic distribution (LE) introduced by Tahmasebi and Rezaei [29], the Weibull distribution (W), and the Lindley distributions(L), whose densities are given by

$$f_{LL}(x;\theta,\beta)=\frac{\theta\beta^2}{(\beta+1)\log(1-\theta)}\frac{(1+x)e^{-\beta x}}{\left[\theta\left(\frac{\beta+1+\beta x}{\beta+1}\right)e^{-\beta x}-1\right]},$$

$$f_{LW}(x;\theta,\beta,\alpha)=\frac{1}{\log(1-\theta)}\frac{\theta\beta\alpha x^{\alpha-1}e^{-\beta x^{\alpha}}}{\theta e^{-\beta x^{\alpha}}-1},$$

$$f_{LE}(x;\theta,\beta)=\frac{1}{\log(1-\theta)}\frac{\theta\beta e^{-\beta x}}{\theta e^{-\beta x}-1},$$

$$f_W(x;\beta,\alpha)=\alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}},$$

$$f_L(x;\beta)=\frac{\beta^2}{\beta+1}(1+x)e^{-\beta x},$$

for $\beta,\alpha,\mathrm{x}>0, 0<\theta<1$, respectively. The first data set represents the number of successive failures for the air-conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes as shown in Proschan [30]. This data consists of 188 observations and has the following values: 1, 1, 3, 3, 3,4, 5, 5, 5, 5, 5, 7, 7, 7, 9, 9, 10, 11, 11, 11, 11, 12, 12, 12, 12, 13, 14, 14, 14, 14, 14, 14, 14, 14, 15, 15, 16, 16, 16, 18, 18, 18, 18, 18, 18, 20, 20, 21, 22, 22, 22, 23, 23, 23, 24, 25, 26, 26, 27, 27, 29, 29, 29, 30, 31, 31, 32, 33, 33, 34, 34, 34, 35, 35, 36, 36, 37, 39, 39, 41, 42, 43, 44, 44, 44, 46, 46, 48, 49, 50, 50, 51, 52, 54, 54, 55, 56, 57, 57, 57, 58, 59, 59, 60, 61, 61, 62, 62, 63, 65, 66, 67, 70, 71, 71, 72, 74, 76, 79, 79, 80, 82, 84, 87, 88, 90, 90, 95, 97, 97, 98, 100, 100, 101, 102, 102, 104, 104, 106, 111, 118, 118, 120, 120, 130, 130, 130, 134, 139, 141, 152, 153, 156, 163, 181, 182, 184, 186, 188, 191, 194, 201, 206, 208, 208, 209, 210, 216, 220, 230, 230, 239, 246, 254, 261, 270, 283, 310, 320, 326, 359, 386, 413, 438, 487, 493, 502, 603.

The second data set consists of 69 observations, which were introduced by Bader and Priest [31] as the tensile strength measurements on 1,000 carbon fiber-impregnated tows at four different gauge lengths. Its values are 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433,3.585, 3.585.

The third data set represents the waiting times (in minutes) before service of 100 bank customers, which was examined and analyzed by Ghitany et al. [32] after fitting the Lindley distributions. The values of this data are 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

For each distribution, we derive the maximum likelihood estimates (MLE), the maximized log likelihood (Log L), the Kolmogorov–Smirnov statistics (K-S) with its respective p-value, the Akaike Information Criterion (AIC), and the Bayesian Information Criterion (BIC). The K-S test is valid to test the goodness of fit of underlying distributions to the failure data, as shown in Bagheri et al. [33]. The results of all data sets are presented in Table 7, Table 8, and Table 9, respectively. The fitted densities and the empirical distribution versus the fitted cdfs for all the data sets are shown in Figures 2, 3, and 4, respectively. They indicate that the PLL distribution fits the data better than the other distributions, except the first data, which was all mostly the same with LL distributions. The KS test statistic takes the smallest value with the largest value of its corresponding p-value for the PLL distribution. Moreover, this conclusion is confirmed from the log likelihood, the AIC, and the BIC for all the fitted models.

Table 7: Parameter estimates, KS statistic, P-value, log likelihood, AIC, and BIC of the air-conditioning system.

Dist.	MLE	K-S	p-value	-log(L)	AIC	BIC
PLL	$\hat{\theta}=0.9907$ $\hat{\beta}=0.0107$ $\hat{\alpha}=0.9727$	0.0362	0.9663	1031.9	2069.8	2079.5
LL	$\hat{\theta}=0.9932$ $\hat{\beta}=0.008897$	0.0359	0.969	1031.9	2067.8	2074.3
LW	$\hat{\theta}=0.9926$ $\hat{\beta}=0.0001$ $\hat{\alpha}=1.6344$	0.0403	0.921	1031.8	2069.6	2079.3
LE	$\hat{\theta}=0.6669$ $\hat{\beta}=0.0082$	0.0523	0.6815	1035.1	2074.2	2080.7
W	$\hat{\beta}=0.0170$ $\hat{\alpha}=0.9110$	0.0562	0.5918	1036.8	2078	2084
L	$\hat{\beta}=0.0215$	0.2134	0.0000	1082.7	2167.3	2170.5

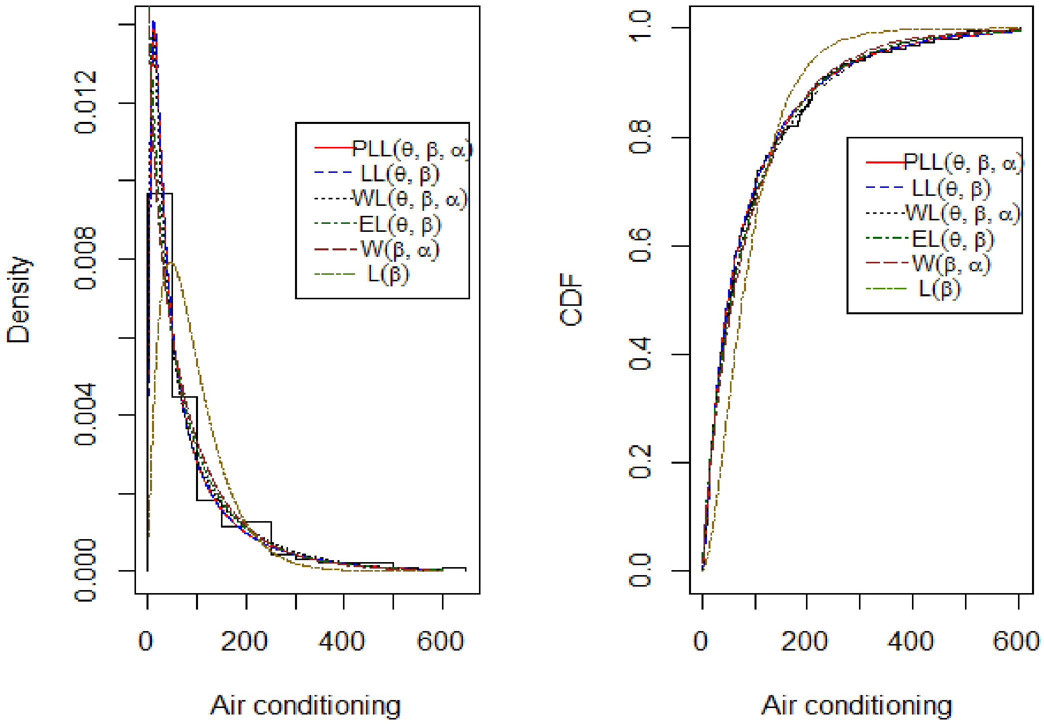

Figure 2. Plots of fitted models of the air-conditioning system data.

Table 8: Parameter estimates, KS statistic, P-value, log likelihood, AIC, and BIC for carbon fiber tensile strength.

Dist.	MLE(std.)	K-S	p-value	-log(L)	AIC	BIC
PLL	$\hat{\theta}=0.6094$ $\hat{\beta}=0.0314$ $\hat{\alpha}=4.1465$	0.0422	0.9993	49	104	110.7
LL	$\hat{\theta}=0.00001$ $\hat{\beta}=0.6545$	0.4011	0.000	119.2	242.4	250.9
LW	$\hat{\theta}=0.8614$ $\hat{\beta}=0.0006$ $\hat{\alpha}=6.9313$	0.0571	0.9684	49.2	104.3	111

LE	$\hat{\beta}=0.000001$ $\hat{\alpha}=0.4079$	0.4483	0.0000	130.7	265.4	269.9
W	$\hat{\beta}=0.0047$ $\hat{\alpha}=5.5049$	0.0563	0.9725	49.6	103.2	107.7
L	$\hat{\beta}=0.6545$	0.4012	0.000	119.2	240.4	242.6

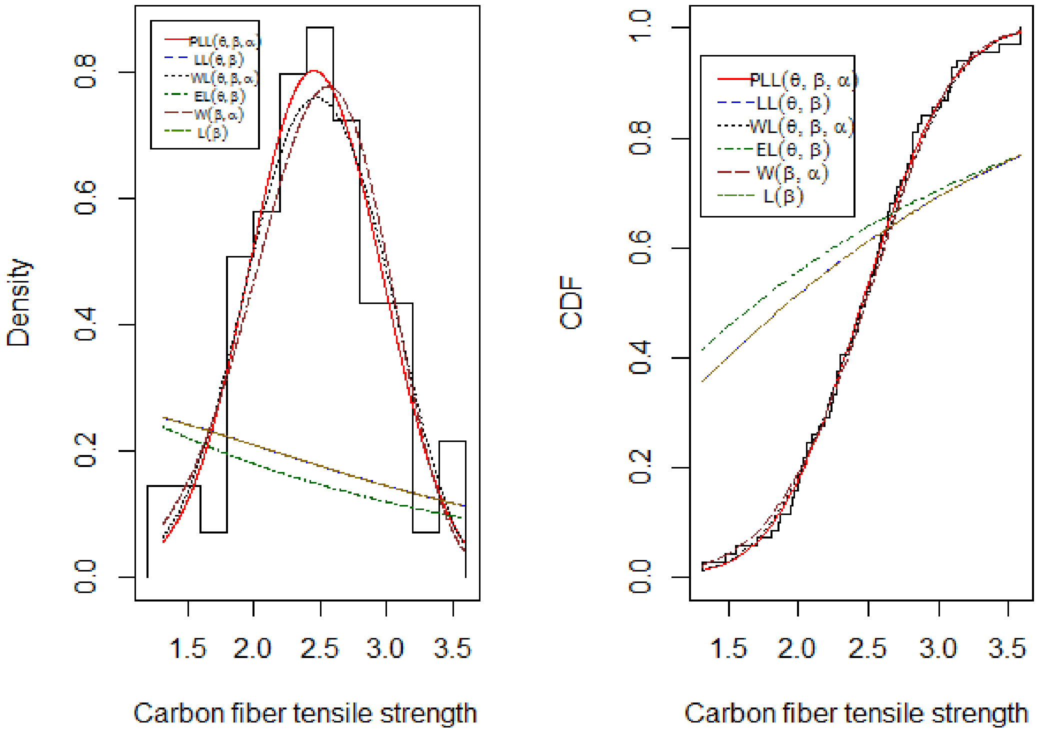

Figure 3. Plots of fitted models of the carbon fiber tensile strength data.

Table 9: Parameter estimates, KS statistics, P-value, log likelihood, AIC, and BIC for the waiting times.

Dist.	MLE	K-S	p-value	-log(L)	AIC	BIC
PLL	$\hat{\theta}=0.9582$ $\hat{\beta}=0.0385$ $\hat{\alpha}=1.3962$	0.0349	0.9997	317.1	640.2	648
LL	$\hat{\theta}=0.00001$ $\hat{\beta}=0.1866$	0.0674	0.7542	319	642	647.2
LW	$\hat{\theta}=0.9544$ $\hat{\beta}=0.0023$ $\hat{\alpha}=2.1039$	0.0496	0.9664	317.4	640.8	648.6
LE	$\hat{\theta}=0.000001$ $\hat{\beta}=0.1012$	0.1739	0.0047	329	662	667.2
W	$\hat{\beta}=0.0305$ $\hat{\alpha}=1.4585$	0.0587	0.8807	318.7	641.4	646.6
L	$\hat{\beta}=0.1866$	0.0686	0.7344	319	640	642.6

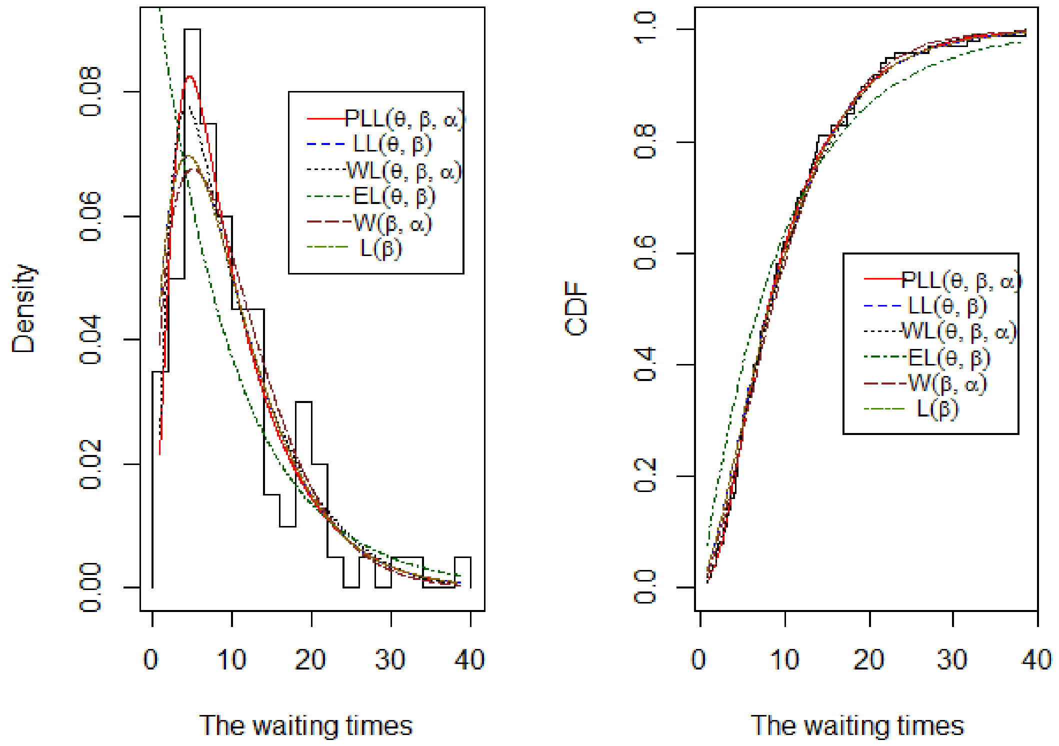

Figure 5. Plots of fitted models of the waiting time data.

9. Concluding remarks

The purpose of this paper was to define a new family of lifetime distributions called the GLPS family of distributions, which generalizes the Lindley power series class of distributions introduced by Liyanage and Pararai [6]. The GLPS class contains some lifetime subclasses and can generates as many useful distributions. The properties of the GLPS class of distributions were derived in flexible and useful forms, including density, survival function, hazard rate function, quantile function, moments, moments generating function, distribution of order statistics, and maximum likelihood estimates. We introduced four subclasses of GLPS distributions in simple and flexible ways for researchers. In addition, we introduced power Lindley logarithmic (PLL) distribution in details to illustrate the befits of the proposed class. Finally, we fitted the PLL distribution to three real data sets and compared it with some existing distributions.

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Conflict of Interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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