King Saud University
College of sciences
Department of mathematics
Second semester 1431/1432 H

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Time: 2 hours

First Midterm Exam Math 580 (Theory Measure I)

Exercise 1:
Let $(B_n)_{n\geq 1}$ be a sequence of subsets of $\mathbb{R}$ defined as :

$$B_{2n-1} =]-2-\frac{1}{n}; 1] \text{ and } B_{2n} = [-1; 2+\frac{1}{n^2}[$$

Determine the superior and inferior limits of (B_n).

Exercise 2:
Determine the $\sigma-$algebra of $\mathbb{R}$ generated by $\mathcal{F} = \{[0;2],[1;3]\}$.

Exercise 3:
Let $\mathcal{A} = \{A \in \mathcal{P}(\mathbb{R}) | A \text{ is countable or } A^c \text{ is uncountable}\}$ be a $\sigma-$algebra on $\mathbb{R}$ and $\mu : \mathcal{A} \longrightarrow [0,\infty]$ a function defined as

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{otherwise.} \end{cases}$$

Prove that μ is a measure on $\mathcal{A}$.

Exercise 4:
Let $(X,\mathcal{A},m)$ be a measure space and $(A_n)_{n\geq 1}$ be a sequence of subsets of X contained in $\mathcal{A}$.

1. Prove that for all $n \geq 1$, $\bigcap_{k\geq n} A_k \in \mathcal{A}$ and $m\left(\bigcap_{k\geq n} A_k\right) \leq \inf_{k\geq n} m(A_k)$.
2. If $B_n := \bigcap_{k\geq n} A_k$ and $b_n := \inf_{k\geq n} m(A_k)$. Prove that $B_n \subseteq B_{n+1}$ and $b_n \leq b_{n+1}$.
3. Deduce that $m\left(\bigcup_n \left(\bigcap_{k\geq n} A_k\right)\right) \leq \sup_n \inf_{k\geq n} m(A_k)$.
4. If we define $\liminf_n A_n = \bigcup_n \left(\bigcap_{k\geq n} A_k\right)$, deduce *Fatou's Lemma* for sequences of subsets:

$$m(\liminf A_n) \leq \liminf m(A_n).$$