الاختبار 35/36 الفصل

س3: 1. f لها مشتقات من الرتبة الثانية متصلة على $\mathbb{R}^2$

هل توجد دالة f تحقق $\frac{\partial f}{\partial x} = 2x+5y$, $\frac{\partial f}{\partial y} = x+3y$

الحل: نفرض أن f موجودة حيث أن f لها مشتقات من الرتبة الثانية متصلة فإن:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

ولدينا $\frac{\partial f}{\partial x} = 2x+5y$, $\frac{\partial f}{\partial y} = x+3y$

فإن

$$\frac{\partial^2 f}{\partial x \partial y} = 5 \quad , \quad \frac{\partial^2 f}{\partial y \partial x} = 1$$

بما أن $\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}$ فإنه لا توجد دالة f لها مشتقات جزئية من الرتبة الثانية متصلة وتحقق

$$\frac{\partial f}{\partial x} = 2x+5y \quad , \quad \frac{\partial f}{\partial y} = x+3y$$

(2) لدينا $u=f(x,y)$ و $x=r\cos\theta$, $y=r\sin\theta$,
و f لها مشتقات من الرتبة الأولى متصلة
حسب قاعدة السلسلة لدينا

$$\begin{cases} \dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x}+\dfrac{\partial u}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial y}+\dfrac{\partial u}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial y} \end{cases}$$

ولدينا:

$$\begin{cases} \dfrac{\partial u}{\partial r}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial r}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial r}=\boxed{\dfrac{\partial u}{\partial x}}\cos\theta+\boxed{\dfrac{\partial u}{\partial y}}\sin\theta \\ \dfrac{\partial u}{\partial \theta}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial \theta}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial \theta}=\boxed{\dfrac{\partial u}{\partial x}}(-r\sin\theta)+\boxed{\dfrac{\partial u}{\partial y}}r\cos\theta \end{cases}$$

النظام 1

فإن

$$r\cos\theta\cdot\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot r\cos^2\theta+\frac{\partial u}{\partial y}r\cos\theta\sin\theta$$

$-$

$$\sin\theta\frac{\partial u}{\partial \theta}=-\frac{\partial u}{\partial x}r\sin^2\theta+\frac{\partial u}{\partial y}r\cos\theta\sin\theta$$

و

وبالتالي:

$$r\cos\theta\cdot\frac{\partial u}{\partial r}-\sin\theta\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\underbrace{(r\cos^2\theta+r\sin^2\theta)}_{r}$$

فإن:

$$\boxed{\frac{\partial u}{\partial x}=\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}}$$

حسب النظام 1 لدينا:

$$r\sin\theta\frac{\partial u}{\partial r}=r\cos\theta\sin\theta\frac{\partial u}{\partial x}+r\sin^2\theta\frac{\partial u}{\partial y}$$

$+$

$$\cos\theta\frac{\partial u}{\partial \theta}=-r\cos\theta\sin\theta\frac{\partial u}{\partial x}+r\cos^2\theta\frac{\partial u}{\partial y}$$

فإن

$$r\sin\theta\frac{\partial u}{\partial r}+\cos\theta\frac{\partial u}{\partial \theta}=r\underbrace{(\sin^2\theta+\cos^2\theta)}_{1}\frac{\partial u}{\partial y}$$

وبالتالي:

$$\boxed{\frac{\partial u}{\partial y}=\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}}$$

(ب) لدينا

$$\begin{cases} \dfrac{\partial u}{\partial x} = \cos\theta \dfrac{\partial u}{\partial r} - \dfrac{\sin\theta}{r} \cdot \dfrac{\partial u}{\partial \theta} \\ \dfrac{\partial u}{\partial y} = \sin\theta \dfrac{\partial u}{\partial r} + \dfrac{\cos\theta}{r} \dfrac{\partial u}{\partial \theta} \end{cases}$$

إذن

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)^2 + \left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)^2$$

$$= \cos^2\theta \left(\frac{\partial u}{\partial r}\right)^2 - 2\cos\theta \frac{\sin\theta}{r} \frac{\partial u}{\partial r}\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

$$+ \sin^2\theta \left(\frac{\partial u}{\partial r}\right)^2 + 2\sin\theta \frac{\cos\theta}{r} \frac{\partial u}{\partial r}\frac{\partial u}{\partial \theta} + \frac{\cos^2\theta}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

وبالتالي:

$$\boxed{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2}$$

ج:

$w=f(x,y)$ لها مشتقات من الرتبة الثانية متصلة

و $x=3u+2v$, $y=6u-4v$

$\frac{\partial^2 w}{\partial u \partial v}=?$

حسب قاعدة السلسلة لدينا:

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}=3\frac{\partial f}{\partial x}+6\frac{\partial f}{\partial y}$$

وحسب قاعدة السلسلة لدينا:

$$\frac{\partial^2 w}{\partial u\partial v}=\frac{\partial\left(\frac{\partial w}{\partial u}\right)}{\partial v}=\frac{\partial}{\partial x}\left(3\frac{\partial f}{\partial x}+6\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial v}+\frac{\partial}{\partial y}\left(3\frac{\partial f}{\partial x}+6\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial v}$$

$$=\left(3\frac{\partial^2 f}{\partial x^2}+6\frac{\partial^2 f}{\partial y\partial x}\right)2+\left(3\frac{\partial^2 f}{\partial x\partial y}+6\frac{\partial^2 f}{\partial y^2}\right)(-4)$$

$$\frac{\partial^2 w}{\partial u\partial v}=6\frac{\partial^2 f}{\partial x^2}+12\frac{\partial^2 f}{\partial y\partial x}-12\frac{\partial^2 f}{\partial x\partial y}-24\frac{\partial^2 f}{\partial y^2}$$

بما أن $w=f(x,y)$ لها مشتقات ثانية جزئية, متصلة فإن

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$$

وبالتالي:

$$\boxed{\frac{\partial^2 w}{\partial u\partial v}=6\frac{\partial^2 f}{\partial x^2}-24\frac{\partial^2 f}{\partial y^2}=6\frac{\partial^2 w}{\partial x^2}-24\frac{\partial^2 w}{\partial y^2}}$$