King Saud University
College of sciences
Department of mathematics
Second semester 1431/1432 H

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Time: 3 hours
Full Marks: 50

Final Exam Math 580 (Theory Measure I)

Exercise 1:(15 points)

Let f_n, f be positives, measurables functions on a measure space $(X, \mathcal{A}, \mu)$ and $f \in L^1$. We suppose that:

$$f \leq \liminf_{n\to+\infty} f_n \quad \mu - a.e \quad \text{and} \ \limsup_{n\to+\infty} \int f_n d\mu \leq \int f d\mu.$$

1. Prove that $\int f d\mu \leq \liminf_{n\to+\infty} \int f_n d\mu$. Deduce that
$$\lim_{n\to+\infty} \int f_n d\mu = \int f d\mu.$$
2. Show that $f = \liminf_{n\to+\infty} f_n \quad \mu - a.e.$
3. Prove that $\forall a, b \in \mathbb{R}$, $|a - b| = a + b - 2\min\{a, b\}$ and deduce that (f_n) converges to f in L^1.
4. Give an example of sequence (f_n) to show that (f_n) does not necessary convergent to $f \quad \mu - a.e.$

Exercise 2:(15 points)

Let $f : [0, 1] \longrightarrow [0, +\infty)$ be a Borel function and $A \subset \mathbb{R}^3$ be the set

$$A := \{(x, y, z)/0 \leq x \leq 1 \ ; \ y^2 + z^2 \leq f(x)\}.$$

1. Prove that the function $F : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $F(x, y, z) = y^2 + z^2 - f(x)$ is a Borel function and deduce that $A \in \mathcal{B}(\mathbb{R}^3)$.
2. For every $x \in [0, 1]$, determine the section $A_x := \{(y, z)/(x, y, z) \in A\}$. Verify that $A_x \in \mathcal{B}(\mathbb{R}^2)$ and compute its Lebesgue measure $\lambda_2(A_x)$.
3. Compute the volume $\lambda_3(A)$ of A as function of f.

4. Verify that $\lambda_3(A) = \frac{\pi}{4}$ if $f(x) = x^3$, $\forall x \in [0,1]$.

Problem:(20 points)

Let $\mathbb{R}_+ = [0,+\infty)$ and for $n \geq 1$, the function $f_n : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{C}$ defined by

$$f_n(t,x,y) := e^{-(n+x)y}\left(e^{iyt} - 1\right).$$

1. Show that $\forall (t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times (0,+\infty)$,

$$\sum_{n\geq 1} |f_n(t,x,y)| \leq |t| \frac{ye^{-y}}{1-e^{-y}}$$

and $f(t,x,y) := \sum_{n=1}^{+\infty} f_n(t,x,y) = \dfrac{e^{-xy}}{e^y - 1}\left(e^{iyt} - 1\right)$.

2. Show that for every $(t,x) \in \mathbb{R} \times \mathbb{R}_+$, the function $y \longmapsto f(t,x,y) \in L^1(\mathbb{R}_+, \lambda)$ and

$$\int_0^{+\infty} f(t,x,y)dy = \sum_{n\geq 1} \frac{it}{(n+x-it)(n+x)}.$$

3. For $(t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$, let $u(t,x,y) := \frac{e^{-xy}}{e^y-1}\sin(yt)$. Show that $\forall (t,x) \in \mathbb{R} \times \mathbb{R}_+$, the function $y \longmapsto u(t,x,y) \in L^1(\mathbb{R}_+, \lambda)$ and

$$\int_0^{+\infty} \frac{e^{-xy}}{e^y - 1}\sin(yt)dy = \sum_{n\geq 1} \frac{t}{(n+x)^2 + t^2}.$$

4. Show that the function $g : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined by

$$g(t,x) := \int_0^{+\infty} \frac{e^{-xy}}{e^y - 1}\sin(yt)dy$$

is of class $C^2(\mathbb{R} \times \mathbb{R}_+)$[1] and satisfying the equation $\frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 g}{\partial x^2} = 0$ on $\mathbb{R} \times \mathbb{R}_+$.

[1]The class $C^2(\mathbb{R} \times \mathbb{R}_+)$ is the set of all functions f twice differentiable and f'' is continuous on $\mathbb{R} \times \mathbb{R}_+$.