King Saud University: Mathematics Department MATH-254
First Semester 1438-39 H Solution of the Final Examination
Maximum Marks = 40 Time: 180 mins.

Name of the Student:——————————— **I.D. No.** ———————————

Name of the Teacher:——————————— **Section No.** ———————————

Note: Check the total number of pages are Five (5).

(**10 Multiple choice questions and Three (3) Full questions**)

The Answer Tables for Q.1 to Q.10 : Marks: 2 for each one ($2 \times 10 = 20$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d										

Quest. No.	Marks
Q. 1 to Q. 10	
Q. 11	
Q. 12	
Q. 13	
Total	

Question 1: The number of iterations required to approximate the root of the equation $2\sin x + x - 1 =$ in $[0, 1]$ accurate to within 10^{-5} using Bisection method is:

(a) 10 **(b)** 13 **(c)** 17 **(d)** 19

Question 2: Newton's iterative method for approximating the square root of a positive number a is given by:

(a) $\left(\frac{x_n}{2} + \frac{a}{x_n^2}\right)$ **(b)** $\left(\frac{x_n}{2} - \frac{a}{x_n^2}\right)$ **(c)** $\frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ **(d)** $\frac{1}{2}\left(x_n - \frac{a}{x_n}\right)$

Question 3: The iterative scheme $x_{n+1} = \dfrac{x_n^2 + 2}{2x_n - 1}$ converges to $\alpha = 2$:

(a) linearly **(b)** quadratically **(c)** cubically **(d)** quartically

Question 4: The condition number of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 - \frac{1}{2n} \end{bmatrix}$ is:

(a) $2n$ **(b)** $4n$ **(c)** $6n$ **(d)** $8n$

Question 5: The norm of the Gauss-Seidel iteration matrix for the linear system of two equations $2x_1 - x_2 = 1,\ x_1 + 2x_2 = 3$ is:

(a) $\frac{1}{2}$ **(b)** $\frac{1}{4}$ **(c)** 1 **(d)** $\frac{1}{8}$

Question 6: If $f(x) = \dfrac{1}{x}$ and $\alpha \neq 1$, then the value of α for which $f[1, \alpha] = f[2, 2, 2]$ is:

(a) -2 **(b)** -4 **(c)** -6 **(d)** -8

Question 7: When using the two-point forward formula with $h = 0.2$ for approximating the value of $f'(1)$, where $f(x) = \ln(x+1)$, we have the computed approximation (accurate to 4 decimal places):

(a) 0.4766 **(b)** 0.4966 **(c)** 0.4666 **(d)** 0.4866

Question 8: If $f(0) = 3, f(1) = \dfrac{\alpha}{2}, f(2) = \alpha$, and Simpson's rule for $\displaystyle\int_0^2 f(x)\,dx$ gives 2, then the value of α is:

(a) 1.0 **(b)** 2.0 **(c)** 0.5 **(d)** 3.0

Question 9: Using data points: $(0, 1), (0.1, 1.1), (0.2, 1.3), (0.3, 1.4), (0.45, 1.5), (0.5, 1.7)$, then the **best** approximate value of $f''(0.3)$ using 3-point difference formula is:

(a) 0.0 **(b)** 0.1 **(c)** 0.2 **(d)** 0.3

Question 10: Given initial-value problem $y' = x + y,\ y(0) = 1$, the approximate value of $y(0.1)$ using Euler's method with $n = 1$ is:

(a) 1.1 **(b)** 1.01 **(c)** 1.02 **(d)** 1.2

Question 11: Find the rate of convergence of the Newton's method at the root $x = 0$ of the equation $x^2e^x = 0$. Use quadratic convergence method to find second approximation to the root using $x_0 = 0.1$. Also, compute the absolute error. [6 points]

Solution. Given $f(x) = x^2e^x$ and so $f'(x) = (x^2 + 2x)e^x$. Using Newton's iterative formula, we get

$$x_{n+1} = x_n - \frac{(x_n^2 e^{x_n})}{((x_n^2 + 2x_n)e_n^x)} = \frac{(x_n + x_n^2)}{(2 + x_n)}, \quad n \geq 0.$$

The fixed point form of the developed Newton's formula is

$$x_{n+1} = g(x_n) = \frac{(x_n + x_n^2)}{(2 + x_n)},$$

where

$$g(x) = \frac{(x + x^2)}{(2 + x)}.$$

By taking derivative, we have

$$g'(x) = \frac{(x^2 + 4x + 2)}{(2 + x)^2},$$

$$g'(0) = \frac{1}{2} = \neq 0.$$

Thus the method converges linearly to the given root.
The quadratic convergent method is modified Newton's method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \qquad n \geq 0,$$

where m is the order of multiplicity of the zero of the function. To find m, we do

$$f''(x) = (x^2 + 4x + 2)e^x, \quad \text{and} \quad f''(0) = 2 \neq 0,$$

so $m = 2$. Thus

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} = x_n - 2\frac{(x_n^2 e^{x_n})}{((x_n^2 + 2x_n)e_n^x)} = x_n - 2\frac{(x_n^2)}{(x_n^2 + 2x_n)}, \qquad n \geq 0.$$

Now using initial approximation $x_0 = 0.1$, we have

$$x_1 = x_0 - 2\frac{(x_0^2)}{(x_0^2 + 2x_0)} = 0.004,$$

and

$$x_2 = x_1 - 2\frac{(x_1^2)}{(x_1^2 + 2x_1)} = 0.0000008,$$

the required two approximations. The possible absolute error is

$$|\alpha - x_2)| = |0.0 - 0.0315| = 0.0000008.$$

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Question 12: Use LU-factorization method with Doolittle's method ($l_{ii} = 1$) to find values of α for which the following linear system has unique solution and infinitely many solutions. Write down the solution for both cases. [7 points]

$$\begin{array}{rcrcrcr} x_1 & + & 0.5x_2 & + & \alpha x_3 & = & 0.5 \\ 2x_1 & - & 3x_2 & + & x_3 & = & -1 \\ -x_1 & - & 1.5x_2 & + & 2.5x_3 & = & -1 \end{array}$$

Solution. We use Simple Gauss-elimination method to convert the following matrix of the given system by using the multiples $m_{21} = 2, m_{31} = -1$ and $m_{32} = 1/4$,

$$A = \begin{pmatrix} 1 & 0.5 & \alpha \\ 2 & -3 & 1 \\ -1 & -1.5 & 2.5 \end{pmatrix},$$

into equivalent an upper-triangular matrix form

$$\begin{pmatrix} 1 & 0.5 & \alpha \\ 0 & -4 & 1-2\alpha \\ 0 & 0 & 0.5\alpha - 0.25 \end{pmatrix},$$

to get LU-factorization of A in the following form

$$A = \begin{pmatrix} 1 & 0.5 & \alpha \\ 2 & -3 & 1 \\ -1 & -1.5 & 2.5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0.25 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0.5 & \alpha \\ 0 & -4 & 1-2\alpha \\ 0 & 0 & 1.5\alpha + 2.25 \end{pmatrix} = LU.$$

Then by solving the lower-triangular system of the form $L\mathbf{y} = [0.5, -1, -1]^T$ and obtained the solution $\mathbf{y} = [0.5, -2, 0]^T$. Now solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ of the form

$$\begin{pmatrix} 1 & 0.5 & \alpha \\ 0 & -4 & 1-2\alpha \\ 0 & 0 & 1.5\alpha + 2.25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -2 \\ 0 \end{pmatrix}.$$

From last equation we have

$$(1.5\alpha + 2.25)x_3 = 0,$$

so for unique solution of the given system $(1.5\alpha + 2.25) \neq 0$ (nonsingular), which implies that $x_3 = 0$. Using backward substitution, we have $x_2 = 0.5$ and $x_1 = 0.25$. Thus, $[0.25, 0.5, 0]^T$ is the unique solution of the given system.

If $(1.5\alpha + 2.25) = 0$ (singular), that is, $\alpha = -1.5$, then for this we must have infinitely many solutions. So to get the infinitely many solutions, we have to solve the following resulting system

$$\begin{array}{rcrcrcr} x_1 & + & 0.5x_2 & + & \alpha x_3 & = & 0.5 \\ & - & 4x_2 & + & (1-2\alpha)x_3 & = & -2 \end{array}$$

By taking $\alpha = -1.5$ and If we choose $x_3 = t \in R,\ t \neq 0$, then we have $x_2 = 0.5 + t$ and $x_1 = 0.25 + t$, so $\mathbf{x}^* = [0.25 + t, 0.5 + t, t]^T$ is the required infinitely many solutions of the given system. •

Question 13: Construct the divided differences table for $f(x) = \ln(x+1)+x^2$ using the values $x = 1, 2, 3, 4, 5$. If the approximation of $f(3.5)$ by a cubic Newton's polynomial is 13.7526, then find the best approximation of $f(3.5)$ by using Newton's polynomial of degree four. Compute the error bound. [7 points]

Solution. The results of the divided differences are listed in Table 1.

Table 1: Divided differences table for $f(x) = \ln(x+1) + x^2$

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
0	1	1.6931				
1	2	5.0986	3.4055			
2	3	10.3863	5.2877	0.9411		
3	4	17.6094	7.2231	0.9677	0.0089	
4	5	26.7918	9.1823	0.9796	0.0040	-0.0012

$$\begin{aligned} f(3.5) \approx p_4(3.5) &= p_3(3.5) + (3.5-1)(3.5-2)(3.5-3)(3.5-4)f[1,2,3,4,5] \\ f(3.5) \approx p_4(3.5) &= 13.7526 + (2.5)(1.5)(0.5)(-1.5)(-0.0012) \\ f(3.5) \approx p_4(3.5) &= 13.7538. \end{aligned}$$

Since the error bound for the fourth-degree polynomial $p_4(x)$ is

$$|f(x) - p_4(x)| = \frac{|f^{(5)}(\eta(x))|}{5!}|(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)|.$$

Taking the fifth derivative of the given function, we have

$$f^{(6)}(x) = \frac{24}{(x+1)^5},$$

and

$$|f^{(5)}(\eta(x))| = M = | \leq \max_{1\leq x\leq 5}\left|\frac{24}{(x+1)^5}\right| = \frac{3}{4} = 0.75,$$

therefore, we get

$$|f(3.5) - p_4(3.5)| \leq \frac{(1.40625)(0.75)}{120} = 8.789 \times 10^{-3},$$

which is the required error bound for the approximation $p_4(3.5)$. •