King Saud University

College of Sciences

Department of Mathematics

Final Examination	Math 244	Semester II	1439-1440	Duration: 3hr.

Calculators are not allowed **2 pages**

Question 1 : [6 pts]

a) Let A be a matrix of order 3 such that $|A| = 3$ and $|A^2 + I| = 2$.
Find $|A + A^{-1}|$.

b) Find the matrix $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ such that $B\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ and $B\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Question 2 : [6 pts]

(a) Let $(V, \langle\ ,\ \rangle)$ be an inner product space.
Compute $\langle 3v_1 - 2v_3, v_1 + v_2 + v_3\rangle$, where $\langle v_1, v_3\rangle = 0$, $\langle v_2, v_3\rangle = 0$, $\langle v_1, v_2\rangle = 2$, $\|v_1\| = 5$ and $\|v_3\| = 2$.

(b) Let $A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 2 & 4 & -1 & 1 & 0 & 2 \end{pmatrix}$.

(i) Find a basis B for the column space of the matrix A.

(ii) Show that B is a basis for $\mathbb{R}^3$.

Question 3 : [7 pts]
Consider the following inner product on $\mathbb{R}^3$:

$$\langle(\mathbf{x}, \mathbf{y}, \mathbf{z}), (\mathbf{x}', \mathbf{y}', \mathbf{z}')\rangle = \mathbf{2xx' + yy' + zz' + xy' + x'y}.$$

Let $u_1 = (-1, 1, x)$, $u_2 = (-1, y, 2)$ and $u_3 = (z, 1, -2)$.

(a) Find the values of x so that $\|u_1\| = 1$.

(b) Find the values of x, y so that $\cos(\theta) = 0$, where θ the angle between u_1 and u_2.

(c) Find the values of x, y, z so that the set $K = \{u_1, u_2, u_3\}$ is orthogonal.

Question 4 : [11 pts]

a) Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation such that $T(1,1,0) = (1,2)$, $T(1,0,1) = (2,1)$ and $T(1,1,1) = (0,0)$.

(i) Prove that $\{(1,1,0),(1,0,1),(1,1,1)\}$ is a basis of $\mathbb{R}^3$.

(ii) Find the expression of $T(x,y,z)$, for $(x,y,z) \in \mathbb{R}^3$.

(iii) Find a basis for $\mathrm{Im}(T)$.

b) Let $B = \{v_1, v_2, v_3, v_4\}$ be a basis for a vector space V, $C = \{u_1 = (1,-1,1),\ u_2 = (1,1,1),\ u_3 = (0,1,1)\}$ a basis of $\mathbb{R}^3$ and $T\colon V \longrightarrow \mathbb{R}^3$ the linear transformation such that

$$[T]_B^C = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

($[T]_B^C$ is the matrix of T with respect to the bases B and C.)

(i) Find $T(v_1), T(v_2), T(v_3), T(v_4)$.

(ii) Find $\mathrm{Rank}(T)$.

(iii) Calculate $\mathrm{nullity}(T)$.

Question 5 : [10 pts]

a) Consider the matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ -2 & -2 & -3 \end{pmatrix}$.

(i) Find $q_A(\lambda) = \det(\lambda I - A)$.

(ii) Deduce that $1, -1, -1$ are the eigenvalues of A.

(iii) Find a matrix P such that $P^{-1}AP$ is a diagonal matrix.

(iv) Find A^{1440} and A^{-1}.

b) For which values of $a \in \mathbb{R}$ the matrix $B = \begin{pmatrix} a & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ is diagonalizable?