King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S1/2019
Full Mark:25. Time 1H30mn
17/10/2019

Question 1[4]. Find and sketch the largest local region of the xy−plane for which the initial value problem

$$\begin{cases} \sqrt{2-\ln(y-3)}dx-(x^2-5x-6)dy=0 \\ y(5)=4, \end{cases}$$

has a unique solution.

Question 2[4+4]. **a)** Solve the initial value problem

$$\begin{cases} x(1+y)dx+(x^2-1+y^2)dy=0, \quad y>-1,\ x\neq 0, \\ y(1)=1. \end{cases}$$

b) Reduce the following equation to a Bernoulli equation and obtain its general solution

$$(1+x^2)^2y\frac{dy}{dx}+2x(1+x^2)y^2-1=0,\ y\neq 0.$$

Question 3[4+4]. **a)** Solve the differential equation

$$\frac{dy}{dx}-e^{-y}\cos(2x)-2=-2\cos^2(x)-e^{-y},\ y\neq 0.$$

b) Find the general solution of the differential equation

$$(2x+y+x\ln x)dx-2xdy=0,\quad x>0.$$

Question 4[5]. The sum of 5000 SAR is invested at a rate of 8% per year. Compounded contiguously. What will be the amount after 25 years.

Complete solutions, Mid - one.

M. 204, First semester 1440/1441

Question 1

$$\begin{cases} y' = \dfrac{dy}{dx} = \dfrac{[2 - \ln(y-3)]^{1/2}}{x^2 - 5x - 6} = f(x,y) \\ y(5) = 4 \end{cases}$$

f is continuous on $R_1 = \{(x,y) ;\ x \neq 6,\ x \neq -1,\ 2 - \ln(y-3) \geq 0 \ \& \ y > 3\}$

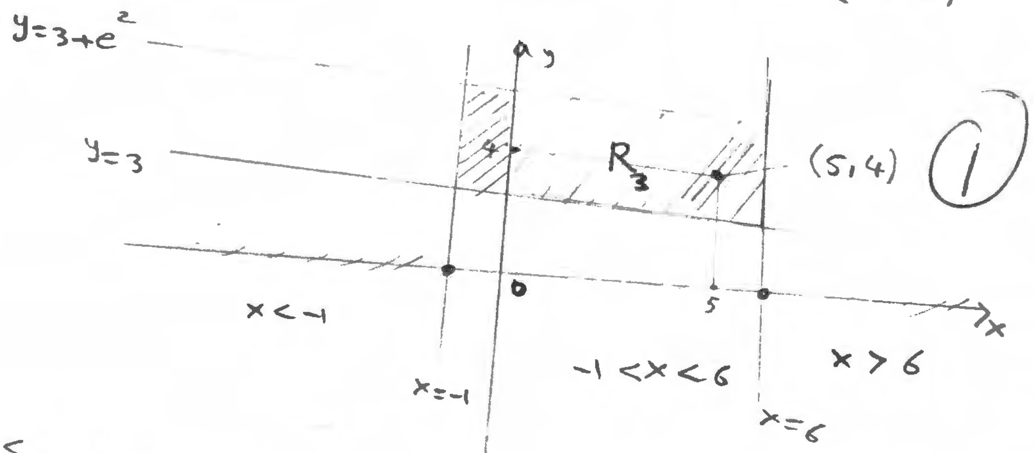
$$\frac{\partial f}{\partial y} = \frac{1}{(x-6)(x+1)} \left(\frac{1}{2}\right) (2 - \ln(y-3))^{-1/2} \frac{(-1)}{y-3}$$ 1

Then f and $\frac{\partial f}{\partial y}$ are continuous on

$R_2 = \{(x,y) ;\ x \neq 6,\ x \neq -1,\ 2 - \ln(y-3) > 0 \text{ and } y > 3\}$

So $\ln(y-3) < 2 \Rightarrow y - 3 < e^2$ or $y < 3 + e^2$ 1

Hence $\boxed{3 < y < 3 + e^2}$, but $5 \in (-1, 6)$



1

So the longest local region of xy-plane for which the IVP has a unique solution is 1

$$R_3 = \{(x,y) ;\ -1 < x < 6,\ 3 < y < 3 + e^2\}$$

Question 2

ⓐ $\begin{cases} x(1+y)dx + (x^2-1+y^2)dy = 0 \ ; & x \neq 0,\ y > -1 \\ y(1) = 1 \end{cases}$

$M = x + xy$, $N = x^2 - 1 + y^2$

$\frac{\partial M}{\partial y} = x$, $\frac{\partial N}{\partial x} = 2x$ $\qquad \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{x}{x(1+y)} = \frac{1}{y+1}$

$\mu(y) = e^{\int \frac{dy}{y+1}} = e^{\ln|y+1|} = y+1.$ 1

Hence

$x(1+y)^2 dx + (x^2-1+y^2)(y+1)dy = 0$ is exact D.E.

Then there exists $F(x,y)$ s.t

$\frac{\partial F}{\partial x} = x(1+y)^2, \quad \frac{\partial F}{\partial y} = (x^2-1+y^2)(y+1)$

So $F(x,y) = \int x(1+y)^2 dx = \frac{1}{2}x^2(1+y)^2 + \phi(y)$

$\frac{\partial F}{\partial y} = \phi'(y) + x^2(1+y) = x^2(y+1) - (y+1) + y^2(y+1)$

$\phi'(y) = y^3 + y^2 - y - 1$

$\phi(y) = \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 - y + c$ 2

Conclusion $F(x,y) = \frac{1}{2}x^2(1+y)^2 + \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 - y + c = 0$

is the solution of the D.E.

But $y(1) = 1 \Rightarrow \frac{1}{2}(4) + \frac{1}{4} + \frac{1}{3} - \frac{1}{2} - 1 + c = 0$

$c = \frac{-13}{12}$ 1

$$\boxed{\frac{1}{2}x^2(1+y)^2 + \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 - y - \frac{13}{12} = 0}$$ is the solution of the IVP

ⓑ $(1+x^2)^2 y \frac{dy}{dx} + 2x(1+x^2)y^2 - 1 = 0$, $y \neq 0$

Then $\left(\begin{array}{l} \frac{dy}{dx} + \frac{2x}{(1+x^2)}y = \frac{1}{(1+x^2)^2}y^{-1} \quad \text{is B. D.Eq},\ n=-1 \\ y y' + \frac{2x}{(1+x^2)}y^2 = \frac{1}{(1+x^2)^2} \end{array} \right.$ 2

We put $u = y^2$, $u' = 2yy'$, $yy' = \frac{u'}{2}$

$u' + \frac{4x}{(1+x^2)} u = \frac{2}{(1+x^2)^2}$ is linear D.E

$\mu(x) = e^{\int \frac{4x\,dx}{1+x^2}} = e^{2\ln(1+x^2)} = (1+x^2)^2$ (1)

$u\mu(x) = u(1+x^2)^2 = \int \frac{2}{(1+x^2)^2} \cdot (1+x^2)^2 dx = \int 2dx$

$u(1+x^2)^2 = 2x + c$, then

$\boxed{y^2(1+x^2)^2 = 2x+c}$ is the solution of the D.E. (1)

Question 3

ⓐ $y' - e^{-y}\cos(2x) - 2 = -2\cos^2(x) - e^{-y}$; $y \neq 0$

$y' - e^{-y}\cos(2x) - 2 = -(1+\cos(2x)) - e^{-y}$ (1)

$y' - e^{-y}\cos(2x) + \cos(2x) = 1 - e^{-y}$

$y' + \cos(2x)(1-e^{-y}) = (1-e^{-y})$

$\frac{dy}{1-e^{-y}} + (\cos(2x) - 1)\,dx = 0$ (1)

$\int \frac{e^y dy}{e^y - 1} + \int (\cos(2x) - 1)\,dx = 0$ (2)

So $\boxed{\ln|e^y - 1| + \frac{1}{2}\sin(2x) - x = c}$ is the solution of the D.E.

ⓑ $(2x + y + x\ln x)\,dx - 2x\,dy = 0$; $x > 0$

$y' - \frac{1}{2x} y = \frac{2+\ln x}{2} = 1 + \frac{1}{2}\ln x$

is Linear D.E (1)

$$\mu(x) = e^{-\int \frac{dx}{2x}} = e^{-\frac{1}{2}\ln x} = e^{\ln x^{-1/2}} = \frac{1}{\sqrt{x}}$$ (1)

$$y\,\mu(x) = \int \frac{1}{\sqrt{x}}\left(1+\frac{1}{2}\ln x\right)dx$$

$$= \int \frac{1}{\sqrt{x}}dx + \frac{1}{2}\int x^{-1/2}\ln x\,dx$$

$$= 2\sqrt{x} + \frac{1}{2}\left[2x^{1/2}\ln x - 4x^{1/2}\right] + c$$ (2)

$$\frac{y}{\sqrt{x}} = 2\sqrt{x} + x^{1/2}\left[\ln x - 2\right] + c$$

$$y = 2x + x(\ln x - 2) + \sqrt{x}\,C$$

$\boxed{y = x\ln x + \sqrt{x}\,C}$ is the G. solution of the D.E

Question 4

Let $y(t)$ be the amount of mony at time t.

$$\frac{\frac{dy}{dt}}{y(t)} = k = \frac{8}{100}, \quad \frac{dy}{y} = \frac{8}{100}dt$$ (1)

Then $y(t) = c\,e^{\left(\frac{8}{100}\right)t}$ (1)

But $y(0) = 5000$, $\Rightarrow y(t) = 5000\,e^{\left(\frac{8}{100}\right)t}$

$$y(25) = 5000\,e^{\left(\frac{8}{100}\right)(25)} = 5000e^{2} \simeq 36,945,28 \text{ SAR}$$ (2)