KING SAUD UNIVERSITY
DEPARTMENT OF MATHEMATICS
TIME: 3H, FULL MARKS: 50, 30/05/2012
MATH 204

Question 1[4,5,6]. **a)** Solve the initial value problem:

$$\begin{cases} (y+\sqrt{xy})dx - xdy = 0, & x>0 \\ y(1)=4. \end{cases}$$

b) Find a suitable integrating factor to convert the differential equation

$$6xydx + (4y + 9x^2)dy = 0,$$

to an exact equation, and then solve it.

c) A radioactive substance has a half life of 5750 years. If 100 grams of this substance is present initially, how much will be left after 1000 years.

Question 2[5,5]. **a)** Find the general solution of the differential equation

$$2x^2y'' + xy' - 3y = 0, \quad x>0,$$

given that $y_1 = 1/x$ is a solution.

b) Solve the differential equation

$$y''' - 4y' = 3x.$$

Question 3. [8] Solve the system of differential equations

$$\begin{cases} \frac{dx}{dt} + 2\frac{dy}{dt} = 2 \\ 2\frac{dx}{dt} = 2y + 3x \end{cases}$$

Question 4. [6] Use power series method to solve the differential equation:

$$y'' + xy' + y = 0.$$

Question 5[5,6]. **a)** Find the Fourier series of the function: $f(x) = 2|x|$, $-\pi < x < \pi$.

b) Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 2, & 0<x<2 \\ -2, & -2<x<0 \\ 0, & |x|>2 \end{cases}$$

and deduce that $\int\limits_0^\infty \frac{\sin\lambda}{\lambda}(1-\cos 2\lambda)d\lambda = \frac{\pi}{2}$.

Solutions:

Q.1. (a) The D. Eq. is homog.

Put $y = ux \Rightarrow dy = u\,dx + x\,du$

Using the values of y and dy in the D. Eq. we get:

$$(\cancel{ux} + x\sqrt{u})\,dx - x(\cancel{u\,dx} + x\,du) = 0$$

$$\Rightarrow x\sqrt{u}\,dx = x^2\,du \Rightarrow \frac{1}{x}\,dx = \frac{1}{\sqrt{u}}\,du$$

$$\Rightarrow \ln x = \frac{1}{2}\sqrt{u} + C$$

or $\ln x = \frac{1}{2}\sqrt{\frac{y}{x}} + C$

(b) $M = 6xy$, $N = 4y + 9x^2$

$\therefore \frac{\partial M}{\partial y} = 6x$, $\frac{\partial N}{\partial x} = 18x$

Since $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{6xy}(6x - 18x) = \frac{-2}{y} = f(y)$

$\therefore$ the integrating factor of this D. Eq. is

$$\mu(y) = e^{-\int \frac{-2}{y}\,dy} = y^2.$$

Multiplying the original D. Eq. by y^2 we get

$$6xy^3\,dx + (4y^3 + 9x^2y^2)\,dy = 0$$

which is an exact D. Eq. $\Rightarrow$ there is a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = 6xy^3 \text{ and } \frac{\partial f}{\partial y} = 4y^3 + 9x^2y^2$$

$$\Rightarrow f(x,y) = \int 6xy^3\,dx = 3x^2y^3 + g(y)$$

But $\frac{\partial f}{\partial y} = 4y^3 + 9x^2y^2$

$\Longrightarrow \quad 9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2$

$\therefore \quad g'(y) = 4y^3 \quad$ or $\quad g(y) = y^4 + c_1$

$\Longrightarrow \quad f(x,y) = 3x^2y^3 + y^4 + c_1$

and the general solution of the exact Eq. is:

$$3x^2y^3 + y^4 + c_1 = c_2$$

or $3x^2y^3 + y^4 = c$

(c) Let $A(t) =: A$ be the amount present at any time t.

$A = A_0 = 100$ gram at $t = 0$

and $A = \frac{1}{2}A_0 = 50$ gram at $t = 5750$ year.

Find the amount left after 1000 year.

We have $A(t) = A_0 e^{kt}$

or $A = 100\, e^{kt}$, using the condition $A = 50$ at $t = 5750$

$\Longrightarrow \quad k = \frac{-\ln 2}{5750} \Longrightarrow A(t) = 100\, e^{\frac{-\ln 2}{5750}t}$.

$\therefore$ At $t = 1000 \Longrightarrow A = 100\, e^{\frac{-\ln 2}{5750}(1000)}$

$\simeq 88.6$ gram.

Q2 (a) Rewrite the D. Eq. in the form

$$y'' + \frac{1}{2x} y' - \frac{3}{2x^2} y = 0 \quad , \quad x > 0$$

Put $P(x) = \frac{1}{2x}$ and $Q(x) = \frac{-3}{2x^2}$

Since $y_1 = \frac{1}{x}$ is a given solution, apply the formula $y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$

to get a second independent solution:

$$y_2 = \frac{1}{x} \int \frac{e^{-\int \frac{1}{2x} dx}}{\left(\frac{1}{x}\right)^2}\,dx = \frac{1}{x}\left(\frac{2}{5} x^{5/2}\right) = \frac{2}{5} x^{3/2}$$

$\therefore$ the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1\left(\frac{1}{x}\right) + c_2\left(x^{\frac{3}{2}}\right).$$

(Or put $y = u y_1$ and use the method of reduction of order).

(b) Homog. Eq. is:

$$y''' - 4y' = 0 \,,$$

hence the auxiliary equation is

$$m^3 - 4m = 0 \Longrightarrow m = 0, \pm 2$$

$$\therefore \; y_c = c_1 + c_2 e^{2x} + c_3 e^{-2x}$$

Since $g(x) = 3x$, the particular solution is of the form

$$y_p = (Ax + B)\,x = Ax^2 + Bx \Longrightarrow$$

$y_p' = 2Ax + B \Rightarrow y_p'' = 2A$ and $y_p''' = 0$

using the values of y_p' and y_p''' in the original D. Eq. give:

$$-8Ax - 4B = x \Rightarrow A = \frac{-1}{8} \text{ and } B = 0.$$

$\therefore y_p = -\frac{1}{8}x^2$, hence the general solution is:

$$y = y_c + y_p$$

$$= C_1 + C_2 e^{2x} + C_3 e^{-2x} - \frac{1}{8}x^2.$$

Q.3 In operator for the system is:

$$\begin{cases} Dx + 2Dy = 2 \\ (2D-3)x - 2y = 0 \end{cases}$$

Apply D to the second equation and add the result to the first equation give:

$$\frac{d^2x}{dt^2} - x = 1 \quad \text{---} (1)$$

$\Rightarrow$ Hom. Eq is: $x'' - x = 0$ and the auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$\therefore X_c = c_1 e^t + c_2 e^{-t}$

Since $g(t) = 1 \Rightarrow X_p = A \Rightarrow X_p' = X_p'' = 0$

using these values in $(1) \Rightarrow A = -1 \Rightarrow X_p = -1$

$\therefore X(t) = X_c + X_p = c_1 e^t + c_2 e^{-t} - 1$

By the second equation in the system we have

$$y = \frac{dx}{dt} - \frac{3}{2}x$$

$$= c_1 e^{t} - c_2 e^{-t} - \frac{3}{2}c_1 e^{t} - \frac{3}{2}c_2 e^{-t} + \frac{3}{2}$$

$$= -\frac{1}{2}c_1 e^{t} - \frac{5}{2}c_2 e^{-t} + \frac{3}{2}.$$

Q.4. Let $y = \sum_{n=0}^{\infty} c_n x^n$

$$\Longrightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2},$$

Using the values of y, y' and y'' in the D. Eq. imply:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Let $k = n-2$ in the first series and $k = n$ in the second and third series imply:

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\Longrightarrow 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k$$

$$+ \sum_{k=1}^{\infty} k c_k x^k$$

$$+ c_0 + \sum_{k=1}^{\infty} c_k x^k = 0$$

$$\Longrightarrow \quad 2C_2 + C_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)C_{k+2} + (k+1)C_k \right] x^k = 0$$

$$\Longrightarrow \quad C_2 = -\frac{1}{2} C_0 \quad \text{and} \quad C_{k+2} = \frac{-1}{k+2} C_k \quad , k = 1, 2, \ldots$$

For $k = 1 \Longrightarrow C_3 = \frac{-1}{3} C_1$

$k = 2 \Longrightarrow C_4 = \frac{-1}{4} C_2 = \frac{1}{8} C_0$

$k = 3 \Longrightarrow C_5 = \frac{-1}{5} C_3 = \frac{1}{15} C_1$

$k = 4 \Longrightarrow C_6 = \frac{-1}{6} C_4 = \frac{-1}{48} C_0$

But $y = \sum_{n=0}^{\infty} C_n x^n$

$$= C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + \ldots$$

$$= C_0 + C_1 x - \frac{1}{2} C_0 x^2 - \frac{1}{3} C_1 x^3 + \frac{1}{8} C_0 x^4$$

$$+ \frac{1}{15} C_1 x^5 - \frac{1}{48} C_0 x^6 + \ldots$$

$$= C_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4 - \frac{1}{48} x^6 + \ldots \right)$$

$$+ C_1 \left(x - \frac{1}{3} x^3 + \frac{1}{15} x^5 - \ldots \right)$$

Q. 5 (a) $\quad f(x) = 2|x| \quad , \quad -\pi < x < \pi$

f is an even function, therefore it has a Fourier cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x \quad , \quad p = \pi$$

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx = \frac{2}{\pi}\int_0^\pi 2x\,dx = 2\pi.$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx = \frac{2}{\pi}\int_0^\pi 2x\cos nx\,dx$$

$$= \frac{4}{\pi}\left[\cancel{\frac{x\sin nx}{n}\Big|_0^\pi} - \int_0^\pi \frac{\sin nx}{n}\,dx\right]$$

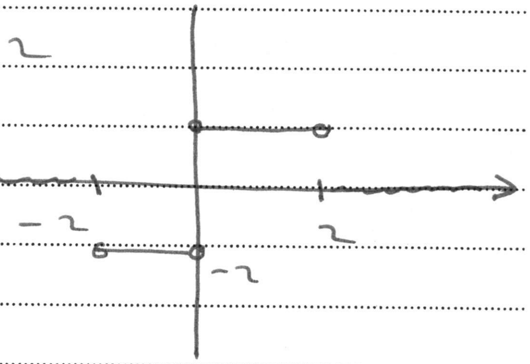
$$= \frac{4}{n^2\pi}\left\{\cos n\pi - 1\right\}$$

$$= \frac{4}{n^2\pi}\left\{(-1)^n - 1\right\}$$

$\therefore$ The Fourier series is

$$f(x) = \pi + \sum_{n=1}^{\infty}\frac{4}{n^2\pi}\left\{(-1)^n - 1\right\}\cos nx$$

(b) $f(x) = \begin{cases} 2, & 0<x<2 \\ -2, & -2<x<0 \\ 0, & |x|>2 \end{cases}$ is an odd function



$\Rightarrow$ The Fourier integral of f on $(-\infty,\infty)$ is the sine integral:

$$f(x) = \frac{2}{\pi}\int_0^\infty B(\alpha)\sin\alpha x\,d\alpha,$$

Where $B(\alpha) = \int_0^\infty f(x)\sin\alpha x\,dx$

$$= \int_0^2 2\sin\alpha x\,dx$$

$$= -\frac{2}{\alpha}\cos\alpha x\Big]_0^2 = \frac{2(1-\cos 2\alpha)}{\alpha}$$

$\therefore$ the Fourier integral of f is:

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{2}{\alpha}(1 - \cos 2\alpha) \sin \alpha x \, d\alpha .$$

At $x = 1$ we have

$$f(1) = 2 = \frac{2}{\pi} \int_0^\infty \frac{2}{\alpha}(1 - \cos 2\alpha) \sin \alpha \, d\alpha$$

or

$$\frac{\pi}{2} = \int (1 - \cos 2\alpha) \frac{\sin \alpha}{\alpha} \, d\alpha .$$