3.8 Harmonic Functions

Let $U \subset \mathbb{R}^n$ be an open set. Let $f : U \to \mathbb{C}$. If f is C^2 on U, and satisfies the Laplace equation

$$\Delta f(x) := \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}(x) = 0, \quad x \in U,$$

then we say that f is a harmonic function on U. The symbol Δ is called the Laplace operator.

In this course, we focus on the case $n = 2$, and identify $\mathbb{R}^2$ with $\mathbb{C}$. The Laplace equation becomes

$$\Delta f(z) = \frac{\partial^2 f}{\partial x^2}(z) + \frac{\partial^2 f}{\partial y^2}(z) = 0, \quad z \in U.$$

Note that a complex function is harmonic if and only if both of its real part and imaginary part are harmonic.

Theorem 3.8.1. *Let f be analytic in an open set $U \subset \mathbb{C}$. Then f is harmonic in U.*

Proof. Let $f = u+iv$. We have seen that f is infinitely many times complex differentiable, which implies that u and v are infinitely many times real differentiable. From the Cauchy-Riemann equation, we get $u_x = v_y$ and $u_y = -v_x$ in U. Thus,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0, \quad v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0,$$

which implies that both u and v are harmonic, and so is f. □

From now on, we assume that a harmonic function is always real valued.

Lemma 3.8.1. *Let u be a real valued C^2 function defined in an open set U. Then u is harmonic in U if and only if $u_x - iu_y$ is analytic in U.*

Proof. Suppose u is harmonic in U. Then $u_x, u_y \in C^1$ and $(u_x)_x = (-u_y)_y$ and $(u_x)y = -(-u_y)_x$. Cauchy-Riemann equation is satisfied by u_x and $-u_y$. So $u_x - iu_y$ is analytic. On the other hand, if $u_x - iu_y$ is analytic, then the Cauchy-Riemann equation implies that $(u_x)_x = (-u_y)_y$, i.e., $u_{xx} + u_{yy} = 0$. So u is harmonic. □

Definition 3.8.1. *Let u be a harmonic function in a domain U. If a real valued function v satisfies that $u + iv$ is analytic in U, then we say that v is a harmonic conjugate of u in U.*

A harmonic conjugate must also be a harmonic function because it is the imaginary part of an analytic function. If v and w are both harmonic conjugates of u in U, then $v_x = -u_y = w_x$ and $v_y = u_x = w_y$ in U. Since U is connected, we get $v - w$ is constant. This means that, the harmonic conjugates of a harmonic function, if it exists, are unique up to an additive constant. Also note that if v is a harmonic conjugate of u, then $-u$ (instead of u) is a harmonic conjugate of v. This is because $-i(u + iv) = v - iu$ is analytic.

Theorem 3.8.2. *Let u be a harmonic function in a simply connected domain U. Then there is a harmonic conjugate of u in U.*

Proof. Let $f = u_x - iu_y$ in U. From the above lemma, f is holomorphic in U. Since U is simply connected, f has a primitive in U, say F. Write $F = \widetilde{u} + i\widetilde{v}$. Then

$$u_x - iu_y = f = F' = \widetilde{u}_x - i\widetilde{u}_y.$$

Thus, $u_x = \widetilde{u}_x$ and $u_y = \widetilde{u}_y$ in U. Since U is connected, we see that $\widetilde{u} - u$ is a real constant. Let $C = \widetilde{u} - u \in \mathbb{R}$. Then $F - C = u + i\widetilde{v}$ is holomorphic in U. Thus, $\widetilde{v}$ is a harmonic conjugate of u. $\square$

Remark. The theorem does not hold if we do not assume that U is simply connected. However, a harmonic conjugate always exists locally: if u is a harmonic function in an open set U, then for any disk $D(z_0, r) \subset U$, there is f, which is analytic in $D(z_0, r)$ and satisfies that $\operatorname{Re} f = u$. Since such f is infinitely many times complex differentiable, we see that u is infinitely many times real differentiable in $D(z_0, r)$. Since $D(z_0, r) \subset U$ can be chosen arbitrarily, we see that every harmonic function is infinitely many times real differentiable.

Example.

1. Let $D = \mathbb{C} \setminus \{0\}$. Let $u(z) = \ln|z| = \frac{1}{2}\ln(x^2 + y^2)$. Then $u_x = \frac{x}{x^2+y^2}$ and $u_y = \frac{y}{x^2+y^2}$. So $u_x - iu_y = \frac{1}{x+iy}$ is holomorphic in D. From the above lemma, u is harmonic. If v is a harmonic conjugate of u in D, then $u + iv$ is a primitive of $u_x - iu_y = \frac{1}{z}$ in D. However, we already know that $\frac{1}{z}$ has no primitive in $\mathbb{C} \setminus \{0\}$. Recall that $\int_{|z|=1} \frac{dz}{z} = 2\pi i \neq 0$. Thus, u has no harmonic conjugates in D.

2. Let $u(x, y) = x^2 + 2xy - y^2$. Then $u_{xx} + u_{yy} = 2 - 2 = 0$. So u is harmonic in $\mathbb{R}^2$. We now find a harmonic conjugate of u. If v is a harmonic conjugate, then $v_y = u_x = 2x + 2y$. Thus, $v = 2xy + y^2 + h(x)$, where $h(x)$ is a differentiable function in x. From $-u_y = v_x$, we get $2y - 2x = 2y + h'(x)$. So we may choose $h(x) = -x^2$. So one harmonic conjugate of u is $2xy + y^2 - x^2$.

Theorem 3.8.3. [**Mean Value Theorem for Harmonic Functions**] *Let u be harmonic on $D(z_0, R)$. Then for any $r \in (0, R)$,*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta;$$

$$u(z_0) = \frac{1}{\pi r^2}\int_{|z-z_0|\le r} u(z)dxdy.$$

Proof. This follows from the Mean Value Theorem for holomorphic functions, and the existence of harmonic conjugates of u in the simply connected domain $D(z_0, R)$. $\square$

Corollary 3.8.1. *With the above setup, if u attains its maximum at z_0, then u is constant in $D(z_0, R)$.*

Proof. We have seen a similar proposition, which says that if f is holomorphic in $D(z_0, R)$, and $|f|$ attains its maximum at z_0, then $|f|$ is constant in $D(z_0, R)$. The two proofs are similar.

Here is another proof. Let f be analytic such that $u = \operatorname{Re} f$. Then e^f is also analytic, and $|e^f| = e^u$. Since u attains its maximum at z_0, $|e^f|$ also attains its maximum at z_0. An earlier proposition shows that $|e^f|$ is constant, which implies that $u = \log |e^f|$ is constant. □

Theorem 3.8.4. **[Maximum Principle for Harmonic Functions]** *Let u be harmonic in a domain U.*

(i) Suppose that u has a local maximum at $z_0 \in U$. Then u is constant.

(ii) If U is bounded, and u is continuous on $\overline{U}$, then there is $z_0 \in \partial U$ such that $u(z_0) = \max\{u(z) : z \in U\}$.

(iii) The above statements also hold if "maximum" is replaced by "minimum".

Proof. (i) From the above corollary, there is $r_0 > 0$ such that u is constant in $D(z_0, r_0)$. Let $w \in U$. Since D is connected, we may find a finite sequence of disks $D_k = D(z_k, r_k)$, $0 \le k \le n$, in U, such that $w \in D_n$ and $D_{k-1} \cap D_k \neq \emptyset$, $1 \le k \le n$. Since each D_k is simply connected, there is f_k holomorphic in D_k such that $u = \operatorname{Re} f_k$ in D_k. We already see that u is constant in D_0. So $\operatorname{Re} f_1 = u$ is constant in $D_0 \cap D_1$. From C-R equations, we see that f_1 is constant in $D_0 \cap D_1$. From the Uniqueness Theorem, we see that f_1 is constant in D_1. Thus, $u = \operatorname{Re} f_1$ is constant in D_1. Using induction, we see that u is constant in every D_k. Since $D_{k-1} \cap D_k \neq \emptyset$, u is constant in $\bigcup_{k=0}^n D_k$. Thus, $f(w) = f(z_0)$ as $w \in D_n$ and $z_0 \in D_0$.

(ii) Since U is bounded, $\overline{U}$ is compact. Since u is continuous on $\overline{U}$, it attains its maximum at some $w_0 \in \overline{U}$. If $w_0 \in \partial U$, we may let $z_0 = w_0$. If $w_0 \in U$, then (i) implies that u is constant in U. The continuity then implies that u is constant in $\overline{U}$. We may take z_0 to by any point on ∂U.

(iii) Note that $-u$ is also harmonic, and when $-u$ attains its maximum, u attains its minimum. □

Corollary 3.8.2. *Suppose u and v are both harmonic in a bounded domain U and continuous on $\overline{U}$. Suppose that $u = v$ on ∂U. Then $u = v$ on $\overline{U}$.*

Proof. Let $h = u - v$. Then h is harmonic in U, continuous on $\overline{U}$, and $h \equiv 0$ on ∂U. From the above theorem, h attains its maximum and minimum at ∂U. So h has to be 0 everywhere, i.e., $u = v$ in $\overline{U}$. □

The above corollary says that, if u is harmonic in a bounded domain U and continuous on $\overline{U}$, then the values of u on U are determined by the values of u on ∂U.

We introduce the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

This mean that, if $f = u + iv$, then

$$f_z := \frac{\partial f}{\partial z} = \frac{1}{2}(u_x + iv_x) - \frac{i}{2}(u_y + iv_y) = \frac{u_x + v_y}{2} + i\frac{v_x - u_y}{2};$$

$$f_{\overline{z}} := \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(u_x + iv_x) + \frac{i}{2}(u_y + iv_y) = \frac{u_x - v_y}{2} + i\frac{v_x + u_y}{2}.$$

So the Cauchy-Riemann equation is equivalent to $f_{\overline{z}} = 0$; and if f is holomorphic, then $f_z = u_x + iv_x = f'$. Moreover, it is clear that

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \frac{1}{4}\Delta.$$

Thus, if f is holomorphic, then $\Delta f = 0$, from which we see again that f is harmonic. If u is harmonic, then from $\partial_{\overline{z}}\partial z u = \frac{1}{4}\Delta u = 0$ we see that $\partial_z u$ is holomorphic, which is used in a proof a theorem.

Remark. The smoothness, mean value theorem and the maximum principle also hold for harmonic functions in R^n for $n \geq 3$. But the technique of complex analysis can not be used. For example, the mean value theorem follows from the divergence theorem.

Homework. Chapter VIII, §1: 7 (a,b,c,e).

1. Find all real-valued C^2 differentiable functions h defined on $(0, \infty)$ such that $u(x, y) = h(x^2 + y^2)$ is harmonic on $\mathbb{C} \setminus \{0\}$.

2. Prove that any positive harmonic function in $\mathbb{R}^2$ is constant. Hint: If f is an entire function with $\operatorname{Re} f > 0$, then consider e^{-f}.
   Remark: This statement does not hold for R^d with $d \geq 3$.

3. Let u be a nonconstant harmonic function on $\mathbb{C}$. Show that for any $c \in \mathbb{R}$, $u^{-1}(c)$ is unbounded. Hint: $\{|z| > R\}$ is connected for any $R > 0$.

3.9 Winding Numbers

Let γ be a closed curve, and $\alpha \in \mathbb{C} \setminus \gamma$. The winding number or index of γ with respect to α is

$$W(\gamma, \alpha) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-\alpha}\, dz.$$

Example. Suppose γ is a Jordan curve. If α lies in the exterior of γ, then applying Cauchy's Theorem to $f(z) = \frac{1}{z-\alpha}$, we get $W(\gamma, \alpha) = 0$. If α lies in the interior of γ, then applying Cauchy's Formula to $f(z) = 1$, we get $W(\gamma, \alpha) = 1$ or -1, where the sign depends on the orientation of γ.

Lemma 3.9.1. *$W(\gamma, \alpha) \in \mathbb{Z}$.*

Proof. Suppose γ is defined on $[a, b]$. Define $F(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s)-\alpha} ds$, $a \le t \le b$. Then F is continuous on $[a, b]$, $F(a) = 0$, $F(b) = 2\pi i W(\gamma, \alpha)$, and $F'(t) = \frac{\gamma'(t)}{\gamma(t)-\alpha}$ for $t \in [a, b]$ other than the partition points, say $a = x_0 < x_1 < \cdots < x_n = b$.We now compute

$$\frac{d}{dt} e^{-F(t)}(\eta(t) - \alpha) = e^{-F(t)}\eta'(t) - e^{-F(t)}F'(t)(\eta(t) - \alpha) = 0, \quad t \in [a, b] \setminus \{x_0, \dots, x_n\}.$$

Hence there is a constant $C \in \mathbb{C}$ such that $C(\eta(t) - \alpha) = e^{F(t)}$, $a \le t \le b$. Since η is closed, we have $e^{F(b)} = e^{F(a)} = e^0 = 1$, which implies that $F(b) \in 2\pi i\mathbb{Z}$. So $W(\gamma, \alpha) = \frac{1}{2\pi i}F(b) \in \mathbb{Z}$. □

Remark. Let θ_0 be an argument of the C in the above proof. From $\eta(t) - \alpha = Ce^{F(t)}$ we see that $\operatorname{Im} F(t)+\theta_0$ is an argument of $\eta(t)-\alpha$ for $a \le t \le b$. Now suppose h is a continuous function on $[a, b]$ such that $h(t)$ is an argument of $\eta(t) - \alpha$ for $a \le t \le b$, then $(h(t) - \operatorname{Im} F(t) - \theta_0)/(2\pi i)$ is an integer-valued continuous function on $[a, b]$, which must be constant. Thus,

$$W(\gamma, \alpha) = \frac{F(b) - F(a)}{2\pi i} = \frac{i \operatorname{Im} F(b) - i \operatorname{Im} F(a)}{2\pi i} = \frac{h(b) - h(a)}{2\pi}.$$

This means that $2\pi W(\gamma, \alpha)$ equals to the total increment of $\arg(z - \alpha)$ along γ.

Lemma 3.9.2. *The map $\alpha \mapsto W(\gamma, \alpha)$ is continuous on $\mathbb{C} \setminus \gamma$.*

Proof. Fix $\alpha_0 \in \mathbb{C} \setminus \gamma$. Let (α_n) be a sequence that converges to α_0. It suffices to show that $\frac{1}{z-\alpha_n} \to \frac{1}{z-\alpha_0}$ uniformly on $z \in \gamma$. Let $r = \operatorname{dist}(\alpha_0, \gamma) > 0$. For n big enough, we have $|\alpha_n - \alpha_0| < r/2$, which implies that $\operatorname{dist}(\alpha_n, \gamma) \ge r/2$. For those n, we have

$$\Big|\frac{1}{z - \alpha_n} - \frac{1}{z - \alpha_0}\Big| = \frac{|\alpha_n - \alpha_0|}{|z - \alpha_n||z - \alpha_0|} \le \frac{|\alpha_n - \alpha_0|}{r^2/2}, \quad z \in \gamma.$$

Thus, $\|\frac{1}{z-\alpha_n} - \frac{1}{z-\alpha_0}\|_\gamma \le \frac{|\alpha_n-\alpha_0|}{r^2/2}$ when n is big enough, which implies that $2\pi i W(\gamma, \alpha_n) = \int_\gamma \frac{1}{z-\alpha_n} dz \to \int_\gamma \frac{1}{z-\alpha_0} dz = 2\pi i W(\gamma, \alpha_0)$. □

Corollary 3.9.1. *$W(\gamma, \cdot)$ is constant on each connected component of $\mathbb{C} \setminus \gamma$.*

Proof. This follows from the above two lemmas and the fact that a continuous integer valued function is constant on a domain. □

Corollary 3.9.2. *$W(\gamma, \alpha) = 0$ if α lies on the unbounded component of $\mathbb{C} \setminus \gamma$.*

Proof. This follows from the fact that, as $\alpha \to \infty$, $\frac{1}{z-\alpha} \to 0$ uniformly in $z \in \gamma$. □

We define a contour γ to be a "sum" of finitely many closed curves γ_k, $1 \le k \le n$, which may or may not have intersections. The repetitions in γ_k's are allowed. The integral along a contour is defined to be $\int_\gamma = \sum_{k=1}^n \int_{\gamma_k}$. The winding number of a contour γ with respect to $\alpha \in \mathbb{C} \setminus \gamma = \mathbb{C} \setminus \bigcup_{k=1}^n \gamma_k$ is $W(\gamma, \alpha) = \sum_{k=1}^n W(\gamma_k, \alpha)$. The above propositions also hold for contours.

Examples.

1. The winding numbers of a trefoil knot in 5 different domains.

Observe that the winding number increases by 1 if we cross the contour from its right to its left; decreases by 1 if we cross the contour from its left to its right.

Theorem 3.9.1. [The General Cauchy's Theorem] *Let f be holomorphic in a domain U. Let γ be a contour in U such that $W(\gamma, \alpha) = 0$ for every $\alpha \in \mathbb{C} \setminus U$. Then $\int_\gamma f = 0$.*

The interested reader may refer to Chapter IV, § 3 of Lang's book for a proof. Note that the condition that $W(\gamma, \alpha) = 0$ for every $\alpha \in \mathbb{C} \setminus U$ is necessary. For otherwise we may construct a counterexample: $f(z) = \frac{1}{z-\alpha}$.

Theorem 3.9.2. [The General Cauchy's Formula] *Let f be holomorphic in a domain U. Let γ be a contour in U such that for every $\alpha \in \mathbb{C} \setminus U$, $W(\gamma, \alpha) = 0$. Let $z_0 \in U$. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, dz = W(\gamma, z_0) f(z_0).$$

Proof. Assuming the general Cauchy's Theorem, the proof of this theorem is not difficult. Let $r > 0$ be such that $\bar{D}(z_0, r) \subset U$. Define a contour η to be $\gamma + (-W(\gamma, z_0))\{|z - z_0| = r\}$. Here if $W(\gamma, z_0) = 0$, then $\eta = \gamma$; if $W(\gamma, z_0) > 0$, this should be understood as $\eta = \gamma + W(\gamma, z_0)\{|z - z_0| = r\}^-$. If Let $U' = U \setminus \{z_0\}$. Then for any $\alpha \in \mathbb{C} \setminus U'$, $W(\eta, \alpha) = 0$. Since $\frac{f(z)}{z-z_0}$ is holomorphic in U', from the general Cauchy's Theorem,

$$0 = \frac{1}{2\pi i} \int_\eta \frac{f(z)}{z - z_0} \, dz = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, dz - \frac{W(\gamma, z_0)}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z - z_0} \, dz$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, dz - W(\gamma, z_0) f(z_0),$$

where the last equality follows from the Cauchy's Formula for Jordan curves. □

Homework. Find the winding numbers for a given closed curve. See the course webpage.