FINAL EXAMINATION, SEM.I, 1429-1430
Department of Mathematics
King Saud University
Math 5701: Topology and Geometry
Time: 3 H   Full Marks: 50

Question # 1.

(a) Let A be a connected subset of a topological space X. If $A \subset B \subset \overline{A}$, then show that B is also connected. If $\overline{A}$ is connected, does it imply A connected? Explain.

(b) Given X a topological space, briefly explain what do you mean by path-components and pathconnectedness of X. Prove or disprove that pathcomponents of X are pathconnected subsets of X.

Question # 2.

(a) Let $g : X \to Z$ be a surjective continuous map and let $X^* = \{g^{-1}(\{z\}) | z \in Z\}$. Give X^* the quotient topology.
(i) If Z is a Hausdorff space, show that X^* is also Hausdorff.
(ii) Show that the map g induces a bijective continuous map $f : X^* \to Z$ which is a homeomporphism if and only if g is a quotient map.

(b) Let $X = I \times I (= [0,1] \times [0,1])$, and define an equivalence relation as follows:
For $(x, y), (u, v) \in X$: $(x, y) \sim (u, v) \Leftrightarrow (x, y) = (u, v)$ Or $(x = 0, u = 1, y = v)$ Or $(x = 1, u = 0, y = v)$. Prove that the quotient space $X/\sim$ is homeomorphic to $S^1 \times I$.

(c) Give an example of a quotient space $X/\sim$ of a Hausdorff space X which is not Hausdorff space.

Or
(c) Define the notion of normal space and give examples. If X is a topological space, A and B are two disjoint closed subsets of X such that there exists a continuous function $f : X \to [0,1]$ with $f(A) = \{0\}$ and $f(B) = \{1\}$, then prove that X is a normal space.

Question # 3.

(a) Define the notion of smooth manifold. Give a detail proof of the fact that S^1 is a 1-dimensional smooth manifold.

(b) Let M be an n-dimensional smooth manifold. If $f : M \to \Re$ is such that there exists a chart (U, φ) around $p \in M$ with $f \circ \varphi^{-1}$ is smooth at $\varphi(p)$. Check whether f is smooth at $p \in M$.

(c) What is Lie group? Discuss that the group $\mathbf{G}l(n, \Re)$ is a Lie group.

Question # 4.

(a) Define the notions of tangent and cotangent spaces.

(b) Let $f : M \to M'$ be smooth between manifolds M and M', and $p \in M$ be such that $df_p : T_p(M) \to T_{f(p)}(M')$ is an isomorphism. **Give a short proof** to show that there exist a neighborhood U of p and V of $f(p)$ such that $f : U \to V$ is a diffeomorphism.

OR

(b) Write down the statement for **Implicit Function Theorem** on smooth manifold (**without proof**), and present one of its applications.

(c) Let M and N be smooth manifolds with M connected, and let $\psi : M \to N$ be smooth. Assume $d\psi \equiv 0$. Prove that ψ is a constant map, that is, $\psi(x) = y_0$ for some $y_0 \in N$ and for all $x \in M$.