2. RANDOM VARIABLES, DISTRIBUTIONS AND EXPECTATIONS

2.1. DISCRETE DISTRIBUTIONS:

Q1. Consider the experiment of flipping a balanced coin three times independently.
Let X= Number of heads – Number of tails.

(a) List the elements of the sample space S.
(b) Assign a value x of X to each sample point.
(c) Find the probability distribution function of X.
(d) Find $P(X \leq 1)$
(e) Find $P(X < 1)$
(f) Find $\mu=E(X)$
(g) Find $\sigma^2=Var(X)$

Q2. It is known that 20% of the people in a certain human population are female. The experiment is to select a committee consisting of two individuals at random. Let X be a random variable giving the number of females in the committee.

1. List the elements of the sample space S.
2. Assign a value x of X to each sample point.
3. Find the probability distribution function of X.
4. Find the probability that there will be at least one female in the committee.
5. Find the probability that there will be at most one female in the committee.
6. Find $\mu=E(X)$
7. Find $\sigma^2=Var(X)$

Q3. A box contains 100 cards; 40 of which are labeled with the number 5 and the other cards are labeled with the number 10. Two cards were selected randomly with replacement and the number appeared on each card was observed. Let X be a random variable giving the total sum of the two numbers.

(i) List the elements of the sample space *S*.
(ii) To each element of *S* assign a value x of X.
(iii) Find the probability mass function (probability distribution function) of X.
(iv) Find P(X=0).
(v) Find P(X>10).
(vi) Find $\mu=E(X)$
(vii) Find $\sigma^2=Var(X)$

Q4. Let X be a random variable with the following probability distribution:

x	–3	6	9
f(x)	0.1	0.5	0.4

1) Find the mean (expected value) of X, $\mu=E(X)$.
2) Find $E(X^2)$.
3) Find the variance of X, Var (X) = σ_X^2.
4) Find the mean of 2X+1, $E(2X+1)=\mu_{2X+1}$.
5) Find the variance of 2X+1, Var(2X+1)= σ_{2X+1}^2.

Q5. Which of the following is a probability distribution function:

(A) $f(x) = \frac{x+1}{10}$; x=0,1,2,3,4 (B) $f(x) = \frac{x-1}{5}$; x=0,1,2,3,4

(C) $f(x) = \frac{1}{5}$; x=0,1,2,3,4 (D) $f(x) = \frac{5-x^2}{6}$; x=0,1,2,3

Q6. Let X be a discrete random variable with the probability distribution function: *f*(x) = *k*x for x=1, 2, and 3.

(i) Find the value of *k*.

(ii) Find the cumulative distribution function (CDF), *F*(x).

(iii) Using the CDF, *F*(x), find P (0.5 < X ≤ 2.5).

Q7. Let X be a random variable with cumulative distribution function (CDF) given by:

$$F(x) = \begin{cases} 0 \ , & x < 0 \\ 0.25, & 0 \le x < 1 \\ 0.6, & 1 \le x < 2 \\ 1 \ , & x \ge 2 \end{cases}$$

(a) Find the probability distribution function of X, f(x).

(b) Find P(1≤X<2). (using both f(x) and F(x))

(c) Find P(X>2). (using both f(x) and F(x))

Q8. Consider the random variable X with the following probability distribution function:

X	0	1	2	3
f(x)	0.4	*c*	0.3	0.1

The value of *c* is

(A) 0.125 (B) 0.2 (C) 0.1 (D) 0.125 (E) -0.2

Q9. The probability distribution for company A is given by:

X	1	2	3
f(x)	0.3	0.4	0.3

and for company B is given by:

Y	0	1	2	3	4
f(y)	0.2	0.1	0.3	0.3	0.1

Show that the variance of the probability distribution for company B is greater than that of company A.

2.2. CONTINUOUS DISTRIBUTIONS:

Q1. If the continuous random variable X has mean μ=16 and variance σ^2=5, then $P(X = 16)$ is

(A) 0.0625 (B) 0.5 (C) 0.0 (D) None of these.

Q2. Consider the probability density function:

$$f(x) = \begin{cases} k\sqrt{x} \ , & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

1) The value of k is:

(A) 1 (B) 0.5 (C) 1.5 (D) 0.667

2) The probability $P(0.3 < X \le 0.6)$ is,

(A) 0.4647 (B) 0.3004 (C) 0.1643 (D) 0.4500

3) The expected value of X, $E(X)$ is,

(A) 0.6 (B) 1.5 (C) 1 (D) 0.667

[Hint: $\int \sqrt{x}\,dx = \frac{x^{3/2}}{(3/2)} + c$]

Q3. If the cumulative distribution function of the random variable X having the form:

$$P(X \le x) = F(x) = \begin{cases} 0 & ; x < 0 \\ x/(x+1) & ; x \ge 0 \end{cases}$$

Then

(1) $P(0 < X < 2)$ equals to

(a) 0.555 (b) 0.333 (c) 0.667 (d) none of these.

(2) If $P(X \le k) = 0.5$, then k equals to

(a) 5 (b) 0.5 (c) 1 (d) 1.5

Q4. For each function below, determine if it can be probability density function. If so, determine c.

a. $f_1(x) = c(2x - x^3)\,;\, for\ 0 < x < \frac{5}{2}$

b. $f_2(x) = c(2x - x^2)\,;\, for\ 0 < x < \frac{5}{2}$

c. $f_3(x) = c(2x^2 - 4x)\,;\, for\ 0 < x < 3$

d. $f_4(x) = c(2x^2 - 4x)\,;\, for\ 0 < x < 2$

Q5. The r.v. X has pdf $f(x) = \begin{cases} c(1 - x^2)\,; & for -1 < x < 1 \\ 0 & ; otherwise \end{cases}$

a. What is the value of c.

b. Find the following probabilities using the pdf of X:

i. P(X<0)

ii. $P\left(X \ge \frac{1}{2}\right)$

iii. $P\left(-\frac{1}{2} < X \le \frac{1}{2}\right)$

iv. P(X>1)

c. Graph the pdf f(x). Show $P\left(X \ge -\frac{1}{2}\right)$ on the graph.

d. What is the cdf of X.

e. Find the probabilities in (b) using the cdf.

Q6. Suppose continuous r.v. X has density function $f(x) = \begin{cases} cx^2\,; & for 1 < x < 2 \\ 0 & ; otherwise \end{cases}$

a. Find the value of the constant c. Graph the pdf.

b. Find $P\left(X \ge \frac{3}{2}\right)$. Show this probability on your graph.

c. Find the cumulative distribution function of X. Graph the cdf.

d. Find $P\left(X \ge \frac{3}{2}\right)$ using the cdf. Show this probability on the cdf graph.

Q7. Prove that $P(a \leq X \leq b) = F(b) - F(a)$ for continuous r.v. X. Explain why the equality signs make no difference.

Q8. For a continuous r.v. X, prove that $P(X \geq c) = 1 - F(c)$.

Q9. A system can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = Cxe^{-x/2}\,; x > 0$$

a. What is the probability that the system functions for at least 5 months.
b. What is the probability that the system functions from 3 to 6 months.
c. What is the probability that the system functions less than 1 month.

Q10. The cumulative distribution function of a continuous r.v. Y is given by

$$F(x) = \begin{cases} 0\,; for \quad y \leq 3 \\ 1 - \dfrac{9}{y^2} \quad ; for \;\; y > 3 \end{cases}$$

Find
a. $P(X \leq 5)$.
b. $P(X > 8)$.
c. the pdf of Y.

Q11. If the density function of the continuous r.v. X is $f(x) = \begin{cases} x\,; 0 < x < 1 \\ 2 - x\,; 1 \leq x < c \\ 0\,;\; o.w. \end{cases}$. Find
a. The value of c.
b. The cumulative distribution function of X.
c. $P(0.8 < X < 0.6c)$.
d. Graph f(x) and F(x). Show the probability in (c) on both graphs.