Mid2 / actu 465

Exercise 1. (2+2+2=6 marks) Let $X_1, \ldots, X_n$ be past claim amounts. Suppose that $X_i|\Theta$ are independent and identically uniformly distributed on the interval $(0, \Theta)$ and Θ is Gamma distributed with parameters α and β.

Determine

a) the hypothetical mean, its mean and variance.

b) the process variance and its mean.

c) the Buhlmann premium.

$X|\theta \sim Unif(0,\theta)$; $\theta \sim Gamma(\alpha,\beta)$

a) $\mu(\theta) = \frac{\theta}{2}$.

$\mu = E\frac{\theta}{2} = \frac{\alpha}{2\beta}$; $v = Var\left(\frac{\theta}{2}\right) = \frac{\alpha}{4\beta^2}$.

b) $v(\theta) = \frac{\theta^2}{12}$; $a = E\left(\frac{\theta^2}{12}\right) = \frac{1}{12}\left(Var(\theta) + E(\theta)^2\right)$

$= \frac{1}{12}\left(\frac{\alpha}{\beta^2} + \frac{\alpha^2}{\beta^2}\right) = \frac{\alpha(1+\alpha)}{12\beta^2}$.

c) $k = \frac{a}{v} = \frac{1+\alpha}{3}$.

$Z = \frac{n}{n+k} = \frac{n}{n + \frac{1+\alpha}{3}} = \frac{3n}{3n+1+\alpha}$.

$P_c = Z\bar{X} + (1-Z)\mu$

$= \frac{3n}{3n+1+\alpha}\bar{X} + \frac{1+\alpha}{4n+1+\alpha} \cdot \frac{\alpha}{2\beta}$.

Exercise 2. (2+2+2=6 marks) Suppose the conditional distribution of the number of claims and the prior distribution are given as follows:

$X\|\Theta$	Probability	Θ	Probability
0	$\Theta/10$	1	0.3
1	$\Theta/5$	2	0.7
2	$1-3\Theta/10$		

Suppose further that a randomly chosen insured has one claim in year 1 and 2 claims in year 2. Determine

a) the hypothetical mean, its mean and variance.
b) the process variance and its mean.
c) the Buhlmann estimate for the number of claims in year 3.

a) $\mu(\theta) = (0)\frac{\theta}{10} + (1)\frac{\theta}{5} + (2)\left(1-\frac{3\theta}{10}\right) = \frac{\theta}{5} + 2 - \frac{3}{5}\theta$

$= \boxed{2 - \frac{2\theta}{5}}$

$\mu = E\left(2 - \frac{2}{5}\theta\right) = 2 - \frac{2}{5}E(\theta)$

$v = var\left(2 - \frac{2}{5}\theta\right) = \frac{4}{25}var(\theta)$

$E(\theta) = (1)(0.3) + (2)(0.7) = 0.3 + 1.4 = 1.7$

$E(\theta^2) = (1^2)(0.3) + (2^2)(0.7) = 0.3 + 2.8 = 3.1$

$var(\theta) = 3.1 - (1.7)^2 = 3.1 - 2.89 = 0.21$

$\rightarrow \mu = 2 - \frac{2}{5}(1.7)$; $v = \frac{4}{25}(0.21) = \boxed{0.033}$

$= \boxed{1.32}$

b) $v(\theta) = (0^2)\frac{\theta}{10} + (1^2)\frac{\theta}{5} + (2^2)\left(1-\frac{3\theta}{10}\right) - \left(2-\frac{2\theta}{5}\right)^2$

$= \frac{\theta}{5} + \cancel{4} - \frac{6}{5}\theta - \cancel{4} - \frac{4}{25}\theta^2 + \frac{8\theta}{5}$

$= \boxed{\frac{3\theta}{5} - \frac{4}{25}\theta^2}$

$a = E\left(\frac{3\theta}{5} - \frac{4}{25}\theta^2\right) = \frac{3}{5}(1.7) - \frac{4}{25}(3.1) = \boxed{0.52}$

c) $k = \frac{a}{v} = 15.75$, $z = \frac{2}{2+k} = 0.11$

$\bar{x} = \frac{1+2}{2} = 1.5$

$P_c = z\bar{x} + (1-z)\mu = (0.11)(1.5) + (0.89)(1.32)$

$= 1.339 \simeq 1.34$

Exercise 3. (2+2+3=7 marks)

You are given:

(i) The number of claims incurred in a year by any insured has a Binomial distribution with parameters m and q.

(ii) The claim frequencies of different insureds are independent.

(iii) The prior distribution M is Geometric with parameter p.

(iv)

Year	Annual Number of insureds	Annual Number of claims
1	120	10
2	100	8
3	180	14
4	200	? 16.28

1) Determine
a) the hypothetical mean, its mean and variance.
b) the process variance and its mean.
2) Suppose $p = q = 0.2$, determine the Buhlmann-Straub credibility estimate of the number of claims in Year 4.

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① $X|M \sim Bin(M,q)$, $M \sim Geo(p)$

a) $\mu(M) = \boxed{qM}$; $\mu = E(qM) = \boxed{q\frac{(1-p)}{p}}$.

$v = Var(qM) = \boxed{q^2\frac{(1-p)}{p^2}}$.

b) $v(M) = \boxed{q(1-q)M}$; $a = E\, q(1-q)M$

$= \boxed{q(1-q)\frac{(1-p)}{p}}$

② $P_c = 200(Z\bar{X} + (1-Z)\mu)$

$= 16.28$

$p = q = 0.2$

$\mu = 0.8$, $v = 0.8$

$a = (0.8)^2 = 0.64$

$k = \frac{a}{v} = 0.8$

$z = \frac{m}{m+k} = \cancel{0.99}\ 0.998$

$\bar{X} = 0.08$

Exercise 4. (2+2+2=6 marks)

You are given total claims for two policyholders:

Policyholder	Year 1	Year 2	Year 3	Year 4	Average
X	730	800	650	700	
Y	655	650	625	750	

Using the nonparametric empirical Bayes method, determine the estimated value of

a) the mean and variance of the hypothetical mean.
b) the mean of the process variance.
c) the Buhlmann credibility premium for Policyholder Y.

a)

b)

$$\bar{x}_1 = \frac{730+800+650+700}{4} = 720$$

$$\bar{x}_2 = \frac{655+650+625+750}{4} = 670$$

$$3933.33 = s_1^2 = \frac{(730-\ )^2+(800-\ )^2+(650-\ )^2+(700-\ )^2}{3}$$

$$3016.66 = s_2^2 = \frac{(655-\ )^2+(650-\ )^2+(625-\ )^2+(750-\ )^2}{3}$$

$$\hat{\mu} = \bar{\bar{x}} = \frac{\cdot+\cdot}{2} = \cancel{666}\ \boxed{695} \quad ; \quad \hat{a} = \frac{\cdot+\cdot}{2} = \cancel{47756.33} = \boxed{3475}$$

$$\hat{v} = \frac{1}{2-1}\left((\bar{x}_1-\bar{x})^2+(\bar{x}_2-\bar{x})^2\right) - \frac{\hat{a}}{4}$$

$$= \cancel{3772.08}\ \boxed{381.25}.$$

c) $P_c = z\bar{x}_2 + (1-z)\hat{\mu}$; $k = \frac{\hat{a}}{\hat{v}} =$

$= (0.3)(670) + (0.7)(695)$ $\quad z = \frac{4}{4+k} = 0.3$

$= \boxed{687.5}$

Bonus Question. (3 marks)
You are given $X_1, \ldots, X_n$ such that:
(i) The model distribution of $X_i|M$ is Poisson with parameter M.
(ii) The prior distribution of M is exponential with parameter δ.
Show that the model satisfies exact credibility.

$$f_{\underline{x},M}(\underline{x},m) = e^{-m}\frac{m^{x_1}}{x_1!}\cdots e^{-m}\frac{m^{x_n}}{x_n!}\delta e^{-\delta m}$$
$$\propto e^{-(n+\delta)m}\, m^{n\bar{x}}$$
$$\pi_{M|\underline{x}}(m|\underline{x}) \propto e^{-(n+\delta)m}\, m^{n\bar{x}} \sim \text{Gamma}(n\bar{x}+1, n+\delta)$$

So

- $P_c = E(X_{n+1}|\underline{X}) = \int E(X_{n+1}|M)\,\pi_M(m|\underline{X})\,dm$

Bayes

$$= \int m\,\pi_M(m|\underline{X})\,dm = \frac{n\bar{X}+1}{n+\delta}$$
$$= \frac{n}{n+\delta}\bar{X} + \frac{\delta}{n+\delta}\frac{1}{\delta}.$$
$$= \cancel{z\bar{X} + (1-z)\mu}$$

$\rightarrow$ ~~Exact Credibility~~.

- $\mu(M) = M \rightarrow \mu = \frac{1}{\delta},\ v = \frac{1}{\delta^2}.$

$v(M) = M \rightarrow a = 1/\delta.$

$k = a/v = \delta,\ z = \frac{n}{n+\delta}.$

Buhlmann

$$P_c = z\bar{X} + (1-z)\mu$$
$$= \frac{n}{n+\delta}\bar{X} + \frac{\delta}{n+\delta}\frac{1}{\delta}.$$

$\Rightarrow$ Exact Credibility.