King Saud University,
College of Sciences
Mathematical Department.

Mid-Term 2/S2/2012
Full Mark:40. Time 1H30mn
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Question 1[8]. Given the initial value problem

$$\left\{\begin{array}{l} x(x^2-4)y''+y=e^x \\ y(-1)=1,\ y'(-1)=0. \end{array}\right. \qquad (*)$$

Find the largest interval for which the initial value problem (*) has a unique solution.

Question 2[7,7]. a) Use reduction of order method to find a second solution of the differential equation

$$xy''-(x+1)y'+y=0,$$

given that $y_1=e^x$ is a solution

b) By using the undetermined coefficients method, give only the form of the particular solution y_p of the differential equation

$$y^{(4)}-y=e^x-xe^{-x}+2\sin x+x^3\cos x$$

Question 3[8]. Solve the differential equation

$$x^2y''-2xy'+2y=2x^3\ln x.$$

Question 4[10]. Solve the following linear system of differential equations

$$\left\{\begin{array}{l} \frac{dx}{dt}=2+4y \\ \frac{dy}{dt}=1+x. \end{array}\right.$$

Solutions:

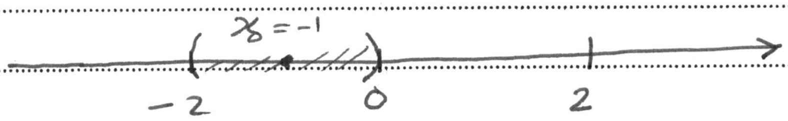
Q.1 $a_2(x) = x(x^2-4)$, $a_1(x) = 0$, $a_0(x) = 1$, $g(x) = e^x$

are continuous functions on $\mathbb{R}$.

and $a_2(x) = 0 \Rightarrow x(x^2-4) = 0 \Rightarrow x = 0, \pm 2$.

Now,

$x_0 = -1$,



lies in the interval $I = (-2, 0)$ on which all a_2, a_1, a_0 and g are cont. and $a_2(x) \neq 0$

$\therefore$ The I.V.P. has a unique solution on

$I = (-2, 0)$.

Q.2 (a) $y_1 = e^x$

Put $y = u y_1 = u e^x$

$\Rightarrow y' = u' e^x + u e^x$

and $y'' = u'' e^x + 2u' e^x + u e^x$

Using the values of y, y' and y'' in the D.E give

$x(u'' e^x + 2u' e^x + \cancel{u e^x}) - (x+1)(u' e^x + \cancel{u e^x}) + \cancel{u e^x} = 0$

$\Rightarrow \quad x e^x u'' + (x-1) e^x u' = 0$, $\div e^x$

$\Rightarrow \quad x u'' + (x-1) u' = 0$

Let $u' = w \rightarrow u'' = w'$

$\Rightarrow \quad x w' + (x-1) w = 0$ (Sep. D.E)

or $\frac{dw}{w} = \frac{-x+1}{x} dx$

$$\Rightarrow \ln|W| = -x + \ln|x| + c$$

$$\Rightarrow W = c_1 x e^{-x} \quad , \quad c_1 = e^{c}$$

$$\Rightarrow u = \int W\,dx = \int c_1 x e^{-x}$$

$$= c_1(-xe^{-x} - e^{-x}) + c_2$$

taking $c_1 = 1$, $c_2 = 0$

$$\Rightarrow y_2 = y_1 u = e^{x}(-xe^{-x} - e^{-x}) = -x - 1$$

Q.2. (b) Charact. eq. is $m^4 - 1 = 0$ ---- $(*)$

$$\Rightarrow (m^2 - 1)(m^2 + 1) = 0 \Rightarrow m = \pm 1, \pm i$$

$$\therefore y_c = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

Let $g_1(x) = e^{x}$, $g_2(x) = -xe^{-x}$ and $g_3(x) = 2\sin x + x^3 \cos x$.

Then $y_{p_1} = (Ae^{x}).x$, since $m = 1$ is a simple root of eq. $(*)$.

$$y_{p_2} = (Bx + C)e^{-x}.x$$

$$= (Bx^2 + Cx)e^{-x}$$ since $m = -1$ is a simple root of eq. $(*)$

and

$$y_{p_3} = \Big[(D_3 x^3 + D_2 x^2 + D_1 x + D_0)\cos x$$

$$+ (F_3 x^3 + F_2 x^2 + F_1 x + F_0)\sin x\Big].x ,$$

since $m = \alpha \pm \beta i = 0 \pm i$ are roots of eq. $(*)$.

$$\therefore y_p = y_{p_1} + y_{p_2} + y_{p_3}$$

Q-3. The D.E is of Cauchy-Euler type.

Hom. Eq: $x^2y'' - 2xy' + 2y = 0$

Put $y = x^m \Longrightarrow$ Charac. Eq. is:

$$m^2 - 3m + 2 = 0 \Longrightarrow m = 1 \text{ or } m = 2$$

$$\Longrightarrow y_c = C_1 x + C_2 x^2$$

By the method of variation of parameters we have:

$$y_p = u_1(x)\, y_1 + u_2(x)\, y_2 ,$$

$$u_1 = \int \frac{W_1}{W}\, dx , \quad u_2 = \int \frac{W_2}{W}\, dx$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$

$$W_1 = \begin{vmatrix} 0 & x^2 \\ 2x\ln x & 2x \end{vmatrix} = -2x^3 \ln x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x\ln x \end{vmatrix} = 2x^2 \ln x$$

$$\therefore u_1 = \int -2x \ln x \, dx = -x^2 \ln x + \frac{x^2}{2}$$

$$u_2 = \int 2\ln x \, dx = 2x\ln x - 2x$$

$$\therefore y_p = u_1 \cdot x + u_2 x^2 = \left(-x^2 \ln x + \frac{x^2}{2}\right) x + (2x\ln x - 2x)\, x^2$$

$$= x^3\left(\ln x - \frac{3}{2}\right)$$

$\therefore$ G. Sol. is $\quad y = y_c + y_p$

Q.4. In operator form the system is:

$$\begin{cases} Dx - 4y = 2 & \text{---(1)} \\ Dy - x = 1 & \text{---(2)} \end{cases}$$

Apply D to eq(2) and add the result to eq (1) to get:

$$D^2 y - 4y = 2 \iff y'' - 4y = 2 \quad \text{---(3)}$$

Hom. eq.: $y'' - 4y = 0 \implies$ Charac. eq is $m^2 - 4 = 0$
$\implies m = \pm 2$

$$\therefore y_c = C_1 e^{2t} + C_2 e^{-2t}$$

Since $g(x) = 2 \implies y_p = A \to y_p' = y_p'' = 0$

$\therefore$ Eq(3) $\implies -4A = 2 \implies A = -\frac{1}{2}$

$$\therefore y_p = -\frac{1}{2}$$

and $y_g = y_c + y_p = C_1 e^{2t} + C_2 e^{-2t} - \frac{1}{2}$

Using y_g in Eq (2) we get:

$$x = Dy - 1$$
$$= (2C_1 e^{2t} - 2C_2 e^{-2t} - 0) - 1$$
$$= 2C_1 e^{2t} - 2C_2 e^{-2t} - 1.$$