King Saud University
College of Sciences
Mathematics Department

MATH204

Exam I, 29/12/1435H
Time: 90 Minutes
Full mark: 25

****

Q.1.[4,4] (a) Find and sketch the largest region in the xy-plane for which the initial value problem

$$\begin{cases} \sqrt{x^2-1}\,\frac{dy}{dx} = 1 + e^x\, ln(y), \\ y(2) = 1, \end{cases}$$

has a unique solution.

(b). Solve the differential equation

$$x\,dy + \left(x\,sec^2\left(\frac{y}{x}\right) - y\right)dx = 0.$$

Q.2.[4,4] (a) Find the general solution of the differential equation

$$\left(3x^2y^2 - 3y^2\right)dx + \left(2x^3y - 6xy + 3y^2\right)dy = 0.$$

(b). Solve the initial value problem

$$\begin{cases} x\,\frac{dy}{dx} + y = \sqrt{x}, \quad x > 0, \\ y(1) = 1. \end{cases}$$

Q.3.[4] Solve the differential equation

$$x\,\frac{dy}{dx} = y\,(x\,y + x + 1), \quad x > 0.$$

Q.4.[5] Find the orthogonal trajectories for the family of curves
$y^2 = e^{x/c}$.

Answer Key

Q.1 (a) $f(x,y) = \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}(1+e^x \ln y)$,

which is cont. on $R_1 = \{(x,y) : |x|>1,\ y>0\}$

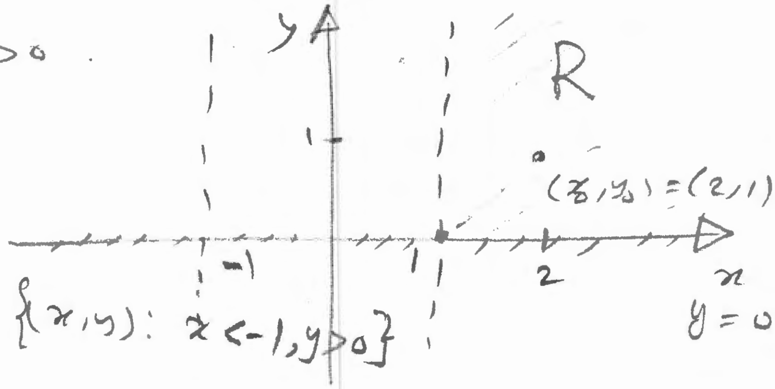
$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{x^2-1}} \cdot \frac{e^x}{y}$, which is cont. on the region

$R_2 = \{(x,y) : |x|>1,\ y\neq 0\}$

Both f and $\frac{\partial f}{\partial y}$ are cont. provided that $|x|>1$ and $y>0$.

i.e. on

$\{(x,y) : x>1,\ y>0\} \cup \{(x,y) : x<-1, y>0\}$



since $(x_0, y_0) = (2,1) \in \{(x,y) : x>1,\ y>0\}$

$\therefore R = \{(x,y) : x>1,\ y>0\}$.

(b) The D.E. can be written as:

$\frac{dy}{dx} = \frac{y}{x} - \sec^2\left(\frac{y}{x}\right)$, hence it is hom. D.E.

$\therefore$ Let $\frac{y}{x} = u \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

Hence we have:

$u + x\frac{du}{dx} = u - \sec^2 u$

$\Rightarrow \frac{1}{\sec^2 u} du = \frac{-1}{x} dx$ or $\cos^2 u \, du = \frac{-1}{x} dx$ Sep. D. E

$\Rightarrow \int \cos^2 u \, du = \int \frac{-1}{x} dx$ or $\int \frac{1}{2}(1+\cos 2u) du = -\ln|x| + c$

$\Rightarrow \frac{1}{2}\left[u + \frac{1}{2}\sin 2u\right] = -\ln|x| + c$

or $\frac{1}{2}\left[\frac{y}{x} + \frac{1}{2}\sin\left(\frac{2y}{x}\right)\right] = -\ln|x| + c$

Q.2 (a) The D.E. is on the standard form: $M dx + N dy = 0$

$M = 3x^2y^2 - 3y^2$, $N = 2x^3y - 6xy + 3y^2$

$\Rightarrow \frac{\partial M}{\partial y} = 6x^2y - 6y$, $\frac{\partial N}{\partial x} = 6x^2y - 6y$

$\therefore$ The D.E. is exact.

$\Rightarrow$ the G.S. is $f(x,y) = c$, where

$f(x,y) = \int M dx = x^3y^2 - 3xy^2 + g(y)$.

Since $\frac{\partial f}{\partial y} = N \Rightarrow \cancel{2x^3y} - \cancel{6xy} + g'(y) = \cancel{2x^3y} - \cancel{6xy} + 3y^2$

$\Rightarrow g'(y) = 3y^2 \Rightarrow g(y) = y^3 + c_1$

$\therefore f(x,y) = x^3y^2 - 3xy^2 + y^3 + c_1$

and the G.S. is

$x^3y^2 - 3xy^2 + y^3 + c_1 = c_2$

or $x^3y^2 - 3xy^2 + y^3 = c$, $c = c_2 - c_1$.

(b) The D.E. $x\frac{dy}{dx}+y=\sqrt{x}$, $x>0$ is 1^{st} order L.D.E which can be written as:

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{1}{\sqrt{x}} \Rightarrow P(x)=\frac{1}{x},\ Q(x)=\frac{1}{\sqrt{x}}.$$

∴ the I.F. is:

$$\mu(x)=e^{\int p(x)\,dx}=e^{\int \frac{1}{x}dx}=x,$$

$$\Rightarrow \frac{d}{dx}(xy)=\sqrt{x},$$

$$\Rightarrow xy=\frac{2}{3}x^{\frac{3}{2}}+c.$$

Since $y(1)=1 \Rightarrow c=\frac{1}{3}$. Therefore the sol. of the i.v.p is: $xy=\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{3}$.

Q.3 The D.E. can be written as:

$$x\frac{dy}{dx}-(x+1)y=xy^2$$, which is Bernoulli's D.E.

$$\Rightarrow \frac{dy}{dx}-\left(1+\frac{1}{x}\right)y=y^2, \quad \div y^2$$

$$\Rightarrow y^{-2}y'-\left(1+\frac{1}{x}\right)y^{-1}=1$$

Let $y^{-1}=w \Rightarrow -y^{-2}\frac{dy}{dx}=\frac{dw}{dx}$

$$\Rightarrow -\frac{dw}{dx}-\left(1+\frac{1}{x}\right)w=1 \quad \text{or}$$

$$\frac{dw}{dx}+\left(1+\frac{1}{x}\right)w=-1,$$ which is L.D.E

with $p(x)=1+\frac{1}{x}$, $Q(x)=-1$

$\Rightarrow$ I.F. is $\mu(x) = e^{\int P(x)\,dx} = e^{\int (1+\frac{1}{x})\,dx} = x e^{x}$

$\Rightarrow \quad \frac{d}{dx}(x e^{x} w) = -x e^{x}$

or $\quad x e^{x} y^{-1} = \int -x e^{x}$

$= -x e^{x} + e^{x} + c$

or $(-x e^{x} + e^{x} + c)\, y = x e^{x}$.

Q.4. Since $y^2 = e^{\frac{x}{c}}$

$\Rightarrow 2 \ln y = \frac{x}{c}$ or $\frac{1}{c} = \frac{2}{x} \ln y$,

differentiating both sides w.r.t x we get

$0 = \frac{2}{x} \frac{1}{y} \frac{dy}{dx} - \frac{2 \ln y}{x^2}$

or $\quad \frac{dy}{dx} = \frac{y \ln y}{x}$

$\Rightarrow$ The D.E. of the orthogonal trajectories is:

$\frac{dy}{dx} = \frac{-x}{y \ln y}$

$\Rightarrow \quad y \ln y \, dy = -x \, dx$

$\therefore \frac{1}{2} y^2 \ln y - \frac{y^2}{4} = -\frac{x^2}{2} + C$

or $\frac{1}{2} y^2 \ln y - \frac{y^2}{4} + \frac{x^2}{2} = C$