J. Math. Anal. Appl. 460 (2018) 496–515

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Criteria of pointwise and uniform directional Lipschitz regularities on tensor products of Schauder functions

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ARTICLE INFO

Article history:
Received 18 January 2016
Available online 2 December 2017
Submitted by P.G. Lemarie-Rieusset

Keywords:
Pointwise and uniform directional Lipschitz regularity
Progressive difference
Tensor product Schauder basis
Multidimensional parameter fractional Wiener field in $\mathbb{R}^d$
Sierpinski cascade function

ABSTRACT

In Ben Slimane and Ben Braiek (2012) and Ben Slimane (2012), we found characterizations of the pointwise and uniform directional regularities of a multi-parameter function in terms of decay rates of either anisotropic Triebel wavelet coefficients or continuous Calderón anisotropic wavelet transform. The purpose of this paper is twofold. We first use a result of Kamont (1996) to provide an easier criterium of uniform directional Lipschitz regularity by decay conditions on the coefficients of the function in a tensor product Schauder basis. As a consequence, we deduce the characterization of the local critical directional Lipschitz regularity. We apply our results for both multidimensional parameter fractional Wiener field in $\mathbb{R}^d$ and Sierpinski cascade function. We then obtain criteria of pointwise directional Lipschitz regularity by decay conditions on either two progressive difference in the given direction or the coefficients of the function in a tensor product Schauder basis.

1. Introduction

The well known pointwise, uniform and local Hölder regularities of a d-parameter function can be characterized by either isotropic wavelet transform or isotropic wavelet (or spline) bases (see [21,28,30]). For $d \geq 2$, such regularities are isotropic and uniform in all directions.

However many images have various anisotropic and directional regularity behaviors which are important for detection of edges, efficient image compression, turbulence, analysis and synthesis of clouds, of bones, or more generally, in medical image processing, for tracking contours (see [3,5,11,12,17,22,29,33,34,36]). Anisotropic behaviors are also important in medical imaging (osteoporosis, muscular tissues, mammographies, etc.), hydrology, fracture surfaces analysis (see [10,11,15,32]).

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https://doi.org/10.1016/j.jmaa.2017.11.061
0022-247X/

A wide range of directional transform ideas have been proposed. Let us cite among them 'Steerable Pyramids' and 'Cortex Transforms' which were developed in the 1980's by vision researchers (Adelson, Freeman, Heeger, and Simoncelli [34] and Watson [36]) to offer increased directional representativeness. Extensions of wavelet bases which can be elongated in particular directions were considered. They include the ridgelets of Candes and Donoho, see [12], or the bandelets of Mallat, see [29], but are efficient with singularities along lines, along hyperplanes, etc, for which wavelets do not deal with efficiently.

An increasing interest in non-isotropic models and semi-elliptic equations has recently turned attention to large classes of anisotropic Besov spaces. Clausel and Vedel [13] studied the sample paths properties of anisotropic selfsimilar Gaussian fields in anisotropic Besov spaces in the case where the anisotropy is a linear mapping of $\mathbb{R}^d$ with eigenvalues having positive real part, in particular, they proved that smoothness in these Besov spaces may be deduced from anisotropy. Anisotropic Besov spaces have played a central role in the mathematical modeling of anisotropic textures. They also have been used to study some PDEs see [35] and for the study of semi-elliptic pseudo-differential operators whose symbols have different degrees of smoothness along different directions see [2].

To our knowledge, the natural definition of pointwise anisotropic regularity which allows for an anisotropic wavelet characterization was first introduced by Ben Slimane [7] in order to investigate the multifractal properties of anisotropic selfsimilar functions. To take into account pointwise directional regularity, Jaffard [22] extended this definition and obtained a characterization by a necessary condition using mixture of anisotropic wavelets and Gabor transform.

The pointwise or uniform Hölder regularity of a d-parameter function in a given direction is the regularity of traces of f taken over 1-dimensional subspace $\mathbb{R} \times \{0\}^{d-1}$, which is a set of vanishing d-dimensional Lebesgue measure. One thought that we cannot characterize pointwise and uniform Lipschitz regularity from d dimensional wavelet coefficients (or transform) of f.

In [33], using curvelet and Hart Smith transforms, Sampo and Sumetkijakan have obtained the pointwise and uniform Hölder regularity on $\mathbb{R}^d$ in a given direction by different necessary and sufficient conditions (due to a parabolic scaling).

But in [9], the relationship between anisotropic regularity and both pointwise and uniform directional Hölder regularities has been established. Full characterizations for pointwise and uniform anisotropic regularities were obtained in:

- [6,9] where Ben Slimane and Ben Braiek have used Triebel anisotropic wavelet bases,
- [7,8], where Ben Slimane has used continuous Calderón anisotropic wavelet transform,
- [1], where Abry, Clausel, Jaffard, Roux and Vedel have used DeVore, Konyagin, and Temlyakov hyperbolic wavelet bases [16].

All these allowed anisotropic criteria to characterize the critical pointwise and uniform Hölder regularity in a given direction (see [6,8,9]).

Note that Calderón wavelet transform (resp. Triebel basis) is tailored to a specific anisotropy that allows different dilations factors related to the fixed anisotropy as opposed to the classical transform (resp. wavelet basis) that relies on a single isotropic dilation factor. On the contrary, hyperbolic wavelet bases are tensor products of one-dimensional wavelets, allowing different dilations factors in all directions. Thus, hyperbolic wavelet bases contain all possible anisotropies. There exist many senses for anisotropy, but here, the sense taken corresponds to different dilations factors.

In this paper, we aim to avoid to pass through anisotropies. We will only focus on Lipschitz regularity (i.e., Hölder regularity less than 1). Note that in signal processing, regularity is often less than 1. In order to study the local properties of a multidimensional parameter fractional Wiener field in $\mathbb{R}^d$, Kamont [25] (resp. [24]) considered some generalized Lipschitz (resp. Hölder) classes described in terms of moduli of smoothness, and characterized these classes by the coefficients of the function in a tensor product Schauder (resp. Franklin) basis. The comprehension of this result is actually the starting point which led us to this paper. We think that, using again the previous Kamont article [24], our techniques can be applied for regularities larger than 1 but we do not pursue this here, because the coefficients in the tensor product of Franklin system of sufficiently high order are complicated (see [24]), since the Franklin system is obtained by the Gram–Schmidt orthonomalization in L^2, of iterated integrations of Schauder functions.

Alternatively, if the studied function f is continuously differentiable of order up to N, then our results remain valid for all derivatives of f of order N.

In section 2, we recall the definitions of pointwise, local, uniform, anisotropic and Lipschitz directional regularities. We provide various comparisons between them and recall main previous results (given in the Lischitz setting). In section 3, we prove a new characterization of the critical uniform directional Lipschitz regularity in terms of decay conditions on the coefficients of the function f itself in a tensor product Schauder basis. We then deduce the characterization of the local directional Lipschitz regularity. In section 4, we apply our results for both the multidimensional parameter fractional Wiener field in $\mathbb{R}^d$ and the Sierpinski cascade function (which can modelize turbulence or cascades). Finally, in section 5, we obtain criteria of pointwise directional Lipschitz regularity by decay conditions on either two progressive difference in the given direction or the coefficients of the function in a tensor product Schauder basis.

2. Definitions, equivalences and previous results

Let d be a positive integer and Ω be a subset of $\mathbb{R}^d$ with non-empty interior. We denote by $C(\Omega)$ the space of continuous d-parameter functions $f : \Omega \to \mathbb{R}$.

Definition 1. Let $0 < \alpha < 1$ and $f \in C(\Omega)$. Let $y \in \Omega$. We say that f is pointwise Lipschitz regular with exponent α at y, denoted by $f \in C^\alpha(y)$, if there exists a positive constant C such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha \quad \forall\, x \in \Omega\,. \tag{1}$$

The critical pointwise Lipschitz regularity of f at y is

$$\alpha_p(y) = \sup\{0 < \alpha < 1 : \quad f \in C^\alpha(y)\}\,. \tag{2}$$

If C in (1) is independent of $y \in \Omega$, then f is uniformly Lipschitz regular with exponent α on Ω, and we write that $f \in C^\alpha(\Omega)$. The critical uniform Lipschitz regularity of f on Ω is

$$\alpha(\Omega) = \sup\{\alpha \in (0, 1) : \quad f \in C^\alpha(\Omega)\}\,. \tag{3}$$

We say that f is uniformly Lipschitz regular on Ω if $\alpha(\Omega) > 0$.

If $N < \alpha < N + 1$, then we say that $f \in C^\alpha(\Omega)$, if all derivatives of f of order N belong to $C^{\alpha - N}(\Omega)$.

We can also define a critical local Lipschitz regularity at a point y through a localization of the critical uniform Lipschitz regularity (see [28]); let $(\Omega_i(y))_{i\in\mathbb{N}}$ be open sets of $\mathbb{R}^d$ such that $\Omega_{i+1}(y) \subset \Omega_i(y)$ and $\bigcap_i \Omega_i(y) = \{y\}$. The critical local Lipschitz regularity of f at y is

$$\alpha_l(y) = \sup \alpha(\Omega_i(y)) = \lim_{i\to\infty} \alpha(\Omega_i(y))\,. \tag{4}$$

It is easy to show that $\alpha_l(y)$ does not depend on the choice of the family $(\Omega_i(y))_{i\in\mathbb{N}}$.

In practice, most methods for estimating the critical pointwise Lipschitz regularity $\alpha_p(y)$ make implicitly or explicitly the assumption that is equal to $\alpha_l(y)$. The domain of validity of this equality has been studied in [28].

The exponent $\alpha_l(y)$ and its evolution in "time" are a relevant tool for characterizing or processing signals (see [27]). Moreover, the critical local Lipschitz regularity is also sensitive to oscillating behavior near the point; if $0 < \alpha < 1$, then for cusp-like singularities, such as $|x-y|^\alpha$, both $\alpha_p(y)$ and $\alpha_l(y)$ coincide and are equal to α, however, for very oscillatory behaviors, such as $|x-y|^\alpha \sin(1/|x-y|^\gamma)$ for $\gamma > 0$, we have $\alpha_p(y) = \alpha$ but $\alpha_l(y) = \alpha/(1+\gamma)$. The latest functions are the most simple examples of chirps at y. In signal analysis, this notion is expected to give a model for functions whose 'instantaneous frequency' increases fast at some time (see [23]).

Let $\psi^{(r)}$, $r = 1, \cdots, 2^d - 1$, be wavelets in $C^\tau(\mathbb{R}^d)$ such that the $2^{dj/2}\psi^{(r)}(2^j x - n)$, $r = 1, \cdots, 2^d - 1$, $j \in \mathbb{Z}$, $n \in \mathbb{Z}^d$, form an orthonormal basis of $L^2(\mathbb{R}^d)$ (see [14,30]). Write

$$f(x) = \sum_{r=1}^{2^d-1} \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}^d} C_{j,k}^{(r)}\, \psi^{(r)}(2^j x - n) \tag{5}$$

where

$$C_{j,n}^{(r)} = 2^{dj} \int_{\mathbb{R}^d} f(x)\, \psi^{(r)}(2^j x - n)\, dx\,. \tag{6}$$

We have the following wavelet characterization of the critical pointwise (resp. uniform Lipschitz regularity of f on $\mathbb{R}^d$) (see [30]);

$$\alpha_p(y) = \min\left(1, \liminf_{j\to\infty} \inf_{E_j} \frac{\log|C_{j,n}^{(r)}|}{\log(2^{-j} + |y - n2^{-j})|}\right) \tag{7}$$

and

$$\alpha(\mathbb{R}^d) = \min\left(1, \liminf_{j\to\infty} \inf_{E_j} \frac{\log|C_{j,n}^{(r)}|}{\log(2^{-j})}\right)\,, \tag{8}$$

where E_j is the set of all (n, r).

If Ω is a bounded open set, then one uses Daubechies wavelets. We have the following wavelet characterization of the critical uniform Lipschitz regularity of f on Ω (see [30]);

$$\alpha(\Omega) = \min\left(1, \liminf_{j\to\infty} \inf_{E_j(\Omega)} \frac{\log|C_{j,n}^{(r)}|}{\log(2^{-j})}\right)\,, \tag{9}$$

where $E_j(\Omega)$ is the set of all (n, r) such that the support of $\psi^{(r)}(2^j x - n)$ is included in Ω. This leads to the following characterization

Proposition 1. *We have*

$$\alpha_l(y) = \min\left(1, \sup_{i\in\mathbb{N}} \liminf_{j\to\infty} \inf_{E_j(\Omega_i(y))} \frac{\log|C_{j,n}^{(r)}|}{\log(2^{-j})}\right)\,. \tag{10}$$

Remark 1. The previous characterizations hold if wavelets are in $C^\tau(\mathbb{R}^d)$ with $\tau > 1$ and the left hand terms in both (8) and (10) are smaller than τ.

We can also replace the above wavelet basis by spline wavelet basis of order 1, i.e., an orthonormal wavelet basis (or a set of two biorthogonal bases) such that ψ is Lipschitz of order 1, for example the Schauder basis. This basis will be recalled in the next section.

If $d \geq 2$ then Definition 1 is isotropic and uniform in all directions. To take into account anisotropic pointwise regularity, Ben Slimane introduced the following definition in [7].

Definition 2. Let $\mathbf{u} = (u_1, \cdots, u_d) \in \mathbb{R}^d$ be such that

$$0 < u_1 \leq \cdots \leq u_d \text{ and } u_1 + \cdots + u_d = d \,. \tag{11}$$

Let $0 < \alpha < u_1$ and $f \in C(\Omega)$. Let $\mathcal{B} = (e_1, \cdots, e_d)$ be an orthonormal basis of $\mathbb{R}^d$. We say that f is $\mathbf{u}$-pointwise Lipschitz regular with exponent h at a point $y \in \Omega$ with respect to (we will write w.r.t.) the basis $\mathcal{B}$ of $\mathbb{R}^d$, denoted by $f \in C^\alpha_{\mathbf{u}}(y, \mathcal{B})$, if there exists a constant $C > 0$ such that

$$|f(y + \sum_{i=1}^{d} t_i e_i) - f(y)| \leq C \sum_{i=1}^{d} |t_i|^{\alpha/u_i} \qquad \forall\, y + \sum_{i=1}^{d} t_i e_i \in \Omega \,. \tag{12}$$

The critical $\mathbf{u}$-pointwise Lipschitz regularity of f at y w.r.t. $\mathcal{B}$ is defined as

$$\alpha_{\mathbf{u},p}(y, \mathcal{B}) = \sup \{\alpha \in (0, u_1) : \quad f \in C^\alpha_{\mathbf{u}}(y, \mathcal{B})\} \,. \tag{13}$$

f is $\mathbf{u}$-uniformly Lipschitz with exponent α on Ω, w.r.t. $\mathcal{B}$, denoted by $f \in C^\alpha_{\mathbf{u}}(\Omega, \mathcal{B})$, if (12) holds for any $y \in \Omega$ with C uniform. The critical $\mathbf{u}$-uniform Lipschitz regularity of f on Ω w.r.t. $\mathcal{B}$ is defined as

$$\alpha_{\mathbf{u}}(\Omega, \mathcal{B}) = \sup \{\alpha \in (0, u_1) : \quad f \in C^\alpha_{\mathbf{u}}(\Omega, \mathcal{B})\} \,. \tag{14}$$

Note that if $\mathbf{u} = (1, \cdots, 1)$, then we return to the isotropic setting.

Full characterizations for $\alpha_{\mathbf{u},p}(y, \mathcal{B})$ and $\alpha_{\mathbf{u}}(\mathbb{R}^d, \mathcal{B})$ were obtained in:

- [6,9] where Ben Slimane and Ben Braiek have used Triebel anisotropic wavelet bases,
- [7], where Ben Slimane has used continuous Calderón anisotropic wavelet transform,
- [1], where Abry, Clausel, Jaffard, Roux and Vedel have used DeVore, Konyagin, and Temlyakov hyperbolic wavelet bases.

Note that Calderón wavelet transform (resp. Triebel basis) is tailored to the specific anisotropy $\mathbf{u}$; roughly speaking it allows dilations factors $1/a, (1/a)^{u_2/u_1}, \cdots, (1/a)^{u_d/u_1}$ (resp. about $2^{[ju_1]}, \cdots, 2^{[ju_d]}$) in directions $e_1, \cdots, e_d$ as opposed to the classical transform (resp. wavelet basis) that relies on the single $1/a$ (resp. 2^j) isotropic dilation factor. On the contrary, hyperbolic wavelet bases are tensor products of one-dimensional wavelets, allowing different dilations factors $2^{j_1}, \cdots, 2^{j_d}$ in directions $e_1, \cdots, e_d$. Thus, hyperbolic wavelet bases contain all possible anisotropies.

To take into pointwise directional pointwise regularity, Jaffard [22] extended Definition 2 in the following way.

Definition 3. Let $f \in C(\Omega)$ and $\overrightarrow{\alpha} = (\alpha_1, \cdots, \alpha_d)$ where $1 > \alpha_1 \geq \cdots \geq \alpha_d > 0$. Let $\mathcal{B} = (e_1, \cdots, e_d)$ be an orthonormal basis of $\mathbb{R}^d$. We say that f is pointwise Hölder Lipschitz with exponent $\overrightarrow{\alpha}$ at a point $y \in \Omega$ w.r.t. $\mathcal{B}$, denoted by $f \in C^{\overrightarrow{\alpha}}(y, \mathcal{B})$, if there exists a constant $C > 0$ such that

$$|f(y + \sum_{i=1}^{d} t_i e_i) - f(y)| \leq C \sum_{i=1}^{d} |t_i|^{\alpha_i} \qquad \forall\, y + \sum_{i=1}^{d} t_i e_i \in \Omega \tag{15}$$

We say that f is uniformly Lipschitz regular with exponent $\overrightarrow{\alpha}$ on Ω, w.r.t. $\mathcal{B}$, denoted by $f \in C^{\overrightarrow{\alpha}}(\Omega, \mathcal{B})$, if (15) holds for all y in Ω with uniform constant C.

Jaffard [22] has characterized $C^{\overrightarrow{\alpha}}(y,\mathcal{B})$ by a necessary condition using mixture of wavelets and Gabor transform.

Remark 2. The last definition can be understood as an anisotropic exponent where the anisotropy is defined as it is usually done for anisotropic functional spaces; let $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_d)$ where $1 > \alpha_1 \geq \cdots \geq \alpha_d > 0$.

Let

$$\frac{1}{\tilde{\alpha}} = \frac{1}{d} \sum_{i=1}^{d} \frac{1}{\alpha_i} . \tag{16}$$

The anisotropy indices for $1 \leq i \leq d$ defined as

$$u_i = \frac{\tilde{\alpha}}{\alpha_i} \tag{17}$$

satisfy (11). It is also clear that $0 < \tilde{\alpha} < u_1$. We have

$$f \in C^{\overrightarrow{\alpha}}(\Omega,\mathcal{B}) \Leftrightarrow f \in C^{\tilde{\alpha}}_{\mathbf{u}}(\Omega,\mathcal{B}) \tag{18}$$

and

$$f \in C^{\overrightarrow{\alpha}}(y,\mathcal{B}) \Leftrightarrow f \in C^{\tilde{\alpha}}_{\mathbf{u}}(y,\mathcal{B}) . \tag{19}$$

The previous definition of Jaffard is connected to the notion of directional Lipschitz regularity.

Definition 4. Let $0 < \alpha < 1$ and $f \in C(\Omega)$. Let e be a fixed vector in the unit sphere S^{d-1}. Let $y \in \Omega$.

We say that f is pointwise Lipschitz regular with exponent α at y, in direction e, denoted by $f \in C^{\alpha}(y,e)$, if there is $C > 0$ such that

$$|f(y+te) - f(y)| \leq C|t|^{\alpha} \quad \forall\, y+te \in \Omega . \tag{20}$$

The critical pointwise Lipschitz regularity of f at y in direction e is

$$\alpha_p(y,e) = \sup\{0 < \alpha < 1 : \quad f \in C^{\alpha}(y,e)\} . \tag{21}$$

If C is independent of $y \in \Omega$, then f is uniformly Hölder Lipschitz with exponent α on Ω in direction e, denoted by $f \in C^{\alpha}(\Omega,e)$. The critical uniform Lipschitz regularity of f on Ω in direction e is

$$\alpha(\Omega,e) = \sup\{\alpha \in (0,1) : \quad f \in C^{\alpha}(\Omega,e)\} . \tag{22}$$

f is uniformly Lipschitz regular on Ω in direction e if $\alpha(\Omega,e) > 0$.

Remark 3. For $y \in \Omega$, let $f_{y,e}$ denotes the 1 variable function $t \mapsto f(y+te)$. Then the critical uniform Lipschitz regularity on Ω in direction e expresses how globally "spiky" the graph of $f_{y,e}$ is, **uniformly on $y \in \Omega$. This makes the use of (8) or (9) very hard since we have to take wavelet coefficients of all functions $f_{y,e}$, where $y \in \Omega$.**

Remark 4. Let e_1 be a fixed vector in the unit sphere S^{d-1}. Let $\mathcal{B}$ denotes any orthonormal basis of $\mathbb{R}^d$ starting with e_1. If f is uniformly Lipschitz regular on Ω then

$$\alpha(y,e_1) = \sup\{\alpha_1 \in (0,1) : \quad \exists \varepsilon > 0 \quad ; \quad f \in C^{\overrightarrow{(\alpha_1,\varepsilon,\cdots,\varepsilon)}}(y,\mathcal{B})\} \tag{23}$$

and

$$\alpha(\Omega,e_1)=\sup\{\alpha_1\in(0,1):\quad \exists\varepsilon>0\quad;\quad f\in C^{\overrightarrow{(\alpha_1,\varepsilon,\cdots,\varepsilon)}}(\Omega,\mathcal{B})\}\ . \tag{24}$$

The lower bound follows from the fact that if $f\in C^{\overrightarrow{(\alpha_1,\varepsilon,\cdots,\varepsilon)}}(y,\mathcal{B})$ (resp. $f\in C^{\overrightarrow{(\alpha_1,\varepsilon,\cdots,\varepsilon)}}(\Omega,\mathcal{B})$) then $f\in C^{\alpha_1}(y,e_1)$ (resp. $f\in C^{\alpha_1}(\Omega,e_1)$), and the upper bound follows from the fact that if $f\in C^{\alpha_1}(y,e_1)$ (resp. $f\in C^{\alpha_1}(\Omega,e_1)$) and $f\in C^{\delta}(\Omega)$ then $f\in C^{\overrightarrow{(\alpha_1,\delta,\cdots,\delta)}}(y,\mathcal{B})$ (resp. $f\in C^{\overrightarrow{(\alpha_1,\delta,\cdots,\delta)}}(\Omega,\mathcal{B})$) because

$$\begin{aligned}|f(y+\sum_{i=1}^d t_ie_i))-f(y)|&\le|f(y+\sum_{i=1}^d t_ie_i))-f(y+t_1e_1)|+|f(y+t_1e_1)-f(y)|\\&\le C|t_1|^{\alpha_1}+C(|t_2|^{\delta}+\cdots+|t_d|^{\delta})\ .\end{aligned}$$

Remark 5. Clearly

$$f\in C^{\overrightarrow{\alpha}}(y,\mathcal{B})\Rightarrow\forall\, i\in\{1,\cdots,d\}\qquad f\in C^{\alpha_i}(y,e_i)\ , \tag{25}$$

and by triangle inequality,

$$f\in C^{\overrightarrow{\alpha}}(\Omega,\mathcal{B})\Leftrightarrow\forall\, i\in\{1,\cdots,d\}\qquad f\in C^{\alpha_i}(\Omega,e_i)\ . \tag{26}$$

Remark 6. In [11], using triangular inequality, Bonami and Estrade have proved that if there exists e_0 such that $0<\alpha(\Omega,e_0)<1$ then the map $e\mapsto\alpha(\Omega,e)$ takes at most d different values. Moreover, it is constant except, perhaps, on the intersection of unit sphere S^{d-1} with a subspace of dimension at most $d-1$ where it may take larger values.

In [33], using curvelet and Hart Smith transforms, Sampo and Sumetkijakan have obtained the pointwise and uniform Lipschitz regularity on $\mathbb{R}^d$ in a given direction e by different necessary and sufficient conditions (due to a parabolic scaling).

In [6,9], Ben Slimane and Ben Braiek used anisotropic Triebel bases to characterize the critical pointwise and uniform Lipschitz regularity in a given direction e.

Using Remark 2, the relationship between critical pointwise (resp. uniform) Lipschitz regularity of the trace of f in direction e_1 at y (resp. on $\mathbb{R}^d$) and anisotropic regularity has been established in [8,9].

Proposition 2. *Let $\mathcal{B}$ denotes any orthonormal basis of $\mathbb{R}^d$ starting with e_1. Let E be the set of all $\mathbf{u}$ satisfying (11) and $u_1=\cdots=u_m$. If f is uniformly Lipschitz on $\mathbb{R}^d$ then*

$$\alpha(y,e_1)=\sup_{\mathbf{u}\in E}\left(\frac{\alpha_{\mathbf{u},p}(y,\mathcal{B})}{u_1}\right) \tag{27}$$

and

$$\alpha(\mathbb{R}^d,e_1)=\sup_{\mathbf{u}\in E}\left(\frac{\alpha_{\mathbf{u}}(\mathbb{R}^d,\mathcal{B})}{u_1}\right)\ . \tag{28}$$

In this paper, we aim to avoid to pass through anisotropies. We will instead characterize both critical pointwise and uniform Lipschitz regularity of the trace of f in the direction e_1 on Ω directly in terms of decay conditions for the coefficients of the function f itself in a tensor product Schauder basis. Our new results are easier than previous ones.

3. Criteria of uniform directional Lipschitz regularity on a tensor product Schauder basis

Since we will use a Kamont result [25], we follow the same notations. Let $I = [0, 1]$ and $f \in C(I^d)$. For $t \in \mathbb{R}$ and $y \in I^d$, define the difference in direction e by the standard formula

$$\Delta_{t,e} f(y) = \begin{cases} f(y+te) - f(y) & \text{if } y+te \in I^d, \\ 0 & \text{if } y+te \notin I^d. \end{cases} \tag{29}$$

Let $e_i = (\delta_{1,i}, \cdots, \delta_{d,i})$ denotes the i-th coordinate vector in $\mathbb{R}^d$. For $f : I^d \to \mathbb{R}$, $t \in \mathbb{R}$, the progressive difference in direction e_i is defined by $\Delta_{t,e_i} f$. From now on, we will write $\Delta_{t,i} f$ instead of $\Delta_{t,e_i} f$.

We need some notations; denote by $\mathcal{D}$ the set $\{1, \cdots, d\}$. Let $A \subset \mathcal{D}$. Let $\boldsymbol{a} = (a_1, \cdots, a_d)$ and $\boldsymbol{b} = (b_1, \cdots, b_d)$ be two vectors of $\mathbb{R}^d$. Put $|\boldsymbol{a}| = |a_1| + \cdots + |a_d|$. Put $\boldsymbol{a}(A) = (\tilde{a}_1, \cdots, \tilde{a}_d)$ where $\tilde{a}_i = a_i$ if $i \in A$, and $\tilde{a}_i = 0$ if $i \notin A$. Write $\boldsymbol{a} \le \boldsymbol{b}$ if $a_i \le b_i$ for all $i \in \mathcal{D}$, and $\boldsymbol{a} < \boldsymbol{b}$ if $a_i < b_i$ for all $i \in \mathcal{D}$. Finally, write $\boldsymbol{a}^{\boldsymbol{b}} = \prod_{i=1}^{d} a_i^{b_i}$. Let also $\mathbf{0}$ and $\mathbf{1}$ denote respectively the vectors $(0, \cdots, 0)$ and $(1, \cdots, 1)$ in $\mathbb{R}^d$.

For $\boldsymbol{h} = (h_1, \cdots, h_d) \in \mathbb{R}^d$ and $A = \{i_1, \cdots, i_k\} \subset \mathcal{D}$ we set

$$\Delta_{\boldsymbol{h},A} f = \Delta_{h_{i_1},i_1} \circ \cdots \circ \Delta_{h_{i_k},i_k} f \ . \tag{30}$$

Clearly

$$\Delta_{h_i,i} \circ \Delta_{h_j,j} f = \Delta_{h_j,j} \circ \Delta_{h_i,i} f \ . \tag{31}$$

For $f \in C(I^d)$, define the moduli of smoothness in the directions A as

$$\omega_A(f, \boldsymbol{t}) = \sup_{\mathbf{0} < \boldsymbol{h} \le \boldsymbol{t}} \sup_{y \in I^d} |\Delta_{\boldsymbol{h},A} f(y)| \qquad \text{for} \quad \boldsymbol{t} \in \mathbb{R}^d, \ \ \mathbf{0} < \boldsymbol{t} \le \mathbf{1} \ . \tag{32}$$

Remark 7. Clearly $f \in C^{\beta_i}(I^d, e_i)$ is equivalent to $\omega_{\{i\}}(f, \boldsymbol{t}) = O(t_i^{\beta_i})$, i.e., there exists $C > 0$ such that $\omega_{\{i\}}(f, \boldsymbol{t}) \le C t_i^{\beta_i}$ when $t_i \to 0$.

Let $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_d)$, with $\mathbf{0} < \boldsymbol{\beta} < \mathbf{1}$. For $\boldsymbol{t} = (t_1, \cdots, t_d)$, define

$$\omega_{\boldsymbol{\beta}}(\boldsymbol{t}) = \prod_{i=1}^{d} t_i^{\beta_i} \ . \tag{33}$$

For a function g given on I^d, $A \subset \mathcal{D}$, and $\boldsymbol{t} \in I^d$, define

$$g(\boldsymbol{t}; A) = g(\boldsymbol{t}(A) + \mathbf{1}(\mathcal{D} \setminus A)) \ . \tag{34}$$

In [25], Kamont considered the following spaces described in terms of moduli of smoothness;

$$Lip(\boldsymbol{\beta}) = \{f \in C(I^d) : \forall (\emptyset \ne A \subset \mathcal{D}) \ \ \omega_A(f, \boldsymbol{t}) = O(\omega_{\boldsymbol{\beta}}(\boldsymbol{t}; A))\} \tag{35}$$

where $O(\boldsymbol{t}(A))$ refers to $\min(t_i : i \in A) \to 0$.

We first show the relationship between these spaces and uniform directional Lipschitz regularity in all directions.

Proposition 3.

1. *If $f \in Lip(\boldsymbol{\beta})$, then $f \in C^{\beta_i}(I^d, e_i)$ for all $i \in \mathcal{D}$.*
2. *Conversely, if $f \in C^{\beta_i}(I^d, e_i)$ for all $i \in \mathcal{D}$, then*

$$\forall\, \mathbf{0} < \boldsymbol{\theta} = (\theta_1, \cdots, \theta_d) \text{ with } |\boldsymbol{\theta}| \le 1 \qquad f \in Lip(\theta_1\beta_1, \cdots, \theta_d\beta_d)\ . \tag{36}$$

Proof of Proposition 3.

1. Let $i \in \mathcal{D}$. If $f \in Lip(\boldsymbol{\beta})$, then $\omega_{\{i\}}(f, \boldsymbol{t}) = O(t_i^{\beta_i})$. The result follows from Remark 7.
2. Conversely, assume that $f \in C^{\beta_i}(I^d, e_i)$ for all $i \in \mathcal{D}$. Let $A \subset \mathcal{D}$ be non-empty. Write $A = \{i_1, \cdots, i_k\}$, we have $\Delta_{h,A} f = \Delta_{h_{i_1}, i_1} g$ where $g = \Delta_{h_{i_2}, i_2} \circ \cdots \circ \Delta_{h_{i_k}, i_k} f$. Since $f \in C^{\beta_{i_1}}(I^d, e_{i_1})$ and g is a linear combination of translated copies of f, then $\omega_A(f, \boldsymbol{t}) = O(t_{i_1}^{\beta_{i_1}})$. Similarly, using property (31), we have $\omega_A(f, \boldsymbol{t}) = O(t_{i_l}^{\beta_{i_l}})$ for all $2 \le l \le k$. On the other hand, since $f \in C(I^d)$ then f is bounded and $\omega_A(f, \boldsymbol{t}) = O(1)$ for all $k+1 \le l \le d$.
   Therefore (36) holds. □

We will first be interested in the characterization of the critical uniform Lipschitz regularity of f on I^d in a given direction e in terms of decay conditions for the coefficients of f in a tensor product Schauder basis. **Without any loss of generality, we can assume that $e = e_1$, because we can take $\mathcal{B}$ starting with e and take coordinates on $\mathcal{B}$.**

Using the partial ordering property

$$Lip(\beta) \subset Lip(\beta') \qquad \forall\, \beta' \le \beta\ , \tag{37}$$

we introduce the following definition as a substitute for $\alpha(I^d, e_1)$

Definition 5. The uniform Lipschitz exponent of f in I^d in the direction e_1 is

$$\widetilde{\alpha}(I^d, e_1) = \sup\left\{\beta_1 \in (0,1) : \exists 0 < \varepsilon < 1 \quad f \in Lip(\beta_1, \varepsilon, \cdots, \varepsilon)\right\}\ . \tag{38}$$

We will show the following proposition

Proposition 4.

1. *If $\alpha(I^d, e_1) = 0$ then $\widetilde{\alpha}(I^d, e_1) = 0$.*
2. *We have always*

$$\widetilde{\alpha}(I^d, e_1) \le \alpha(I^d, e_1)\ . \tag{39}$$

3. *If f is uniform Lipschitz on I^d then $\widetilde{\alpha}(I^d, e_1) = \alpha(I^d, e_1)$.*

Proof of Proposition 4.

1. The first result is a consequence of the first part of Proposition 3.
2. The second result follows from the second part of Proposition 3.
3. Assume that $f \in C^{\delta}(I^d)$ for $0 < \delta < 1$. Clearly $f \in C^{\delta}(I^d, e_i)$ for all $i \in \mathcal{D}$ and $\alpha(I^d, e_1) \ge \delta$. Let $\beta_1 < \alpha(I^d, e_1)$. Since $f \in C^{\beta}(I^d, e_1)$, then the second result in Proposition 3 implies that $f \in$

$Lip((1-(d-1)\theta)\beta_1, \theta\delta, \cdots, \theta\delta)$ for all $0 < \theta \leq \dfrac{1}{d-1}$. Therefore by letting θ tends to 0, we obtain $\widetilde{\alpha}(I^d, e_1) \geq \alpha(I^d, e_1)$. □

In [24,25], Kamont characterized the space $Lip(\beta)$ in terms of decay conditions for the coefficients of f in a tensor product Schauder basis.

Let $\{\phi_k, k \geq 0\}$ be the following Schauder functions on I, i.e. $\phi_0 = 1$, $\phi_1(t) = t$, and for $k \geq 2$, $k = 2^j + n$ with $j \geq 0$ and $1 \leq n \leq 2^j$, $\phi_k(t) = \phi(2^{j+1}t - 2n + 1)$ (with support $[(n-1)2^{-j}, n2^{-j}]$), where $\phi(t) = \max(0, 1-|t|)$ (the so-called Schauder function).

In $\mathbb{R}^d$, with $d \geq 2$, we consider the family $\{\Phi_{\boldsymbol{k}}, \boldsymbol{k} \geq 0\}$ of tensor products of Schauder functions, i.e. $\Phi_{\boldsymbol{k}}(x) = \phi_{k_1}(x_1) \cdots \phi_{k_d}(x_d)$ for $\boldsymbol{k} = (k_1, \cdots, k_d)$.

For $j \in M = \{-2, -1, 0, 1, 2, \cdots\}$, let

$$\tilde{N}_{-2} = \{0\},\ \tilde{N}_{-1} = \{1\},\ \text{and}\ \tilde{N}_j = \{2^j + n : n = 1, \cdots, 2^j\}\ \text{ for } j \geq 0 \tag{40}$$

and for a vector $\boldsymbol{j} = (j_1, \cdots, j_d)$ we put

$$\tilde{N}_{\boldsymbol{j}} = \tilde{N}_{j_1} \times \cdots \tilde{N}_{j_d}\ . \tag{41}$$

Let for $f \in C(I^d)$, $i \in \mathcal{D}$, $x \in I^d$ and $k \geq 0$

$$c_{i,0}(f)(x) = f(x - x_i e_i)\ ,\ c_{i,1}(f)(x) = f(x + (1 - x_i)e_i) - f(x - x_i e_i)\ , \tag{42}$$

and for $k = 2^j + n \in \tilde{N}_j$ with $j \geq 0$

$$c_{i,k}(f)(x) = f(x + (\frac{2n-1}{2^{j+1}} - x_i)e_i) - \frac{1}{2}(f(x + (\frac{n-1}{2^j} - x_i)e_i) + f(x + (\frac{n}{2^j} - x_i)e_i)). \tag{43}$$

For $\boldsymbol{k} = (k_1, \cdots, k_d)$ we put

$$C_{\boldsymbol{k}}(f) = c_{1,k_1} \circ \cdots \circ c_{d,k_d}(f)\ . \tag{44}$$

Then for any $f \in C(I^d)$ we have

$$f = \sum_{\boldsymbol{j} \in M^d} \sum_{\boldsymbol{k} \in \tilde{N}_{\boldsymbol{j}}} C_{\boldsymbol{k}}(f) \Phi_{\boldsymbol{k}}\ . \tag{45}$$

In $\sum_{\boldsymbol{j} \in M^d}$ we assume the following order: for $\boldsymbol{j} = (j_l, ..., j_d)$ and $\boldsymbol{j}' = (j'_l, ..., j'_d)$, if $\max(j_1, ..., j_d) < \max(j'_1, ..., j'_d)$, then $\boldsymbol{j}$ precedes $\boldsymbol{j}'$.

For f given by (45) we put

$$\tau_{\boldsymbol{j}}(f) = \sup_{\boldsymbol{k} \in \tilde{N}_{\boldsymbol{j}}} |C_{\boldsymbol{k}}(f)|\ . \tag{46}$$

The following wavelet characterization of spaces $Lip(\boldsymbol{\beta})$ is due to Kamont [25]:

Proposition 5. *Let*

$$\boldsymbol{t}_{\boldsymbol{j}} = (2^{-\max(j_1,0)}, \cdots, 2^{-\max(j_d,0)})\ . \tag{47}$$

Then, for $\boldsymbol{0} < \boldsymbol{\beta} < \boldsymbol{1}$,

$$f \in Lip(\boldsymbol{\beta}) \quad \Leftrightarrow \quad \tau_{\boldsymbol{j}}(f) = O(\omega_{\boldsymbol{\beta}}(\boldsymbol{t}_{\boldsymbol{j}}))\ \ as\ \ |\boldsymbol{j}| \to \infty\ . \tag{48}$$

Thanks to Proposition 4, the last result leads to the following characterization

Theorem 1. *If f is uniform Lipschitz on I^d, then*

$$\alpha(I^d, e_1) = \sup\left\{\beta_1 \in (0,1) : \exists 0 < \varepsilon < 1 \quad \tau_{\boldsymbol{j}}(f) = O(\omega_{(\beta_1,\varepsilon,\cdots,\varepsilon)}(\boldsymbol{t_j}))\ \right\}$$
$$= \min\left(1, \liminf_{|\boldsymbol{j}|\to\infty} \frac{\log \tau_{\boldsymbol{j}}(f)}{\log(2^{-\max(j_1,0)})}\right) .$$

Analogously, for $1 \le i \le d$

$$\alpha(I^d, e_i) = \min\left(1, \liminf_{|\boldsymbol{j}|\to\infty} \frac{\log \tau_{\boldsymbol{j}}(f)}{\log(2^{-\max(j_i,0)})}\right) . \tag{49}$$

Using result (26), we also deduce that $f \in C^{\overrightarrow{\alpha}}(I^d, \mathcal{B})$ for all $\overrightarrow{\alpha}$ with $\alpha_i < \min\left(1, \liminf_{|\boldsymbol{j}|\to\infty} \frac{\log \tau_{\boldsymbol{j}}(f)}{\log(2^{-\max(j_i,0)})}\right)$ for all $i \in \{1, \cdots, d\}$.

We can define a critical local directional Lipschitz regularity at a point y through a localization of the critical uniform directional Lipschitz regularity; let $(\Omega_i(y))_{i\in\mathbb{N}}$ be open sets of $\mathbb{R}^d$ such that $\Omega_{i+1}(y) \subset \Omega_i(y)$ and $\bigcap_i \Omega_i(y) = \{y\}$. The critical directional local Lipschitz regularity of f at y, in direction e_1 is

$$\alpha_l(y, e_1) = \sup_{i\in\mathbb{N}} \alpha(\Omega_i(y)) = \lim_{i\to\infty} \alpha(\Omega_i(y), e_1) . \tag{50}$$

As in Proposition 1 and Remark 1, we have the following characterization:

Proposition 6. *If f is uniform Lipschitz on I^d, then*

$$\alpha_l(y, e_1) = \min\left(1, \sup_{i\in\mathbb{N}} \liminf_{|\boldsymbol{j}|\to\infty} \inf_{E_{\boldsymbol{j}}(\Omega_i(y))} \frac{\log |C_{\boldsymbol{k}}(f)|}{\log(2^{-\max(j_1,0)})}\right) , \tag{51}$$

where $E_{\boldsymbol{j}}(\Omega_i(y))$ is the set of all $\boldsymbol{k} \in \tilde{N}_{\boldsymbol{j}}$ such that the support of $\Phi_{\boldsymbol{k}}$ is included in $\Omega_i(y)$.

4. Examples

4.1. The multidimensional parameter fractional Wiener field in $\mathbb{R}^d$

The fractional anisotropic Wiener field with the multidimensional parameter $\alpha = (\alpha_1, \cdots, \alpha_d)$, with $0 < \alpha_i < 2$, is a Gaussian field $\{B^{(\alpha)}(t) : t \in \mathbb{R}^d\}$, with continuous realizations, $EB^{(\alpha)}(t) = 0$, and the covariance kernel

$$EB^{(\alpha)}(t)B^{(\alpha)}(s) = K_\alpha(t,s) , \quad \text{where } K_\alpha = K_{\alpha_1} \times \cdots \times K_{\alpha_d}$$

and K_{α_i}, is the covariance kernel of one-dimensional fractional Brownian motion with parameter α, i.e.,

$$K_{\alpha_i}(t,s) = \frac{1}{2}(|t|^{\alpha_i} + |s|^{\alpha_i} - |t-s|^{\alpha_i}) .$$

As for (33), for $\mathbf{0} < \boldsymbol{\beta} = (\beta_1, \cdots, \beta_d) < \mathbf{1}$ and $\mathbf{0} < \boldsymbol{t} = (t_1, \cdots, t_d) < \mathbf{1}$, let

$$\omega_{\boldsymbol{\beta},\frac{1}{2}}(\boldsymbol{t}) = (\prod_{i=1}^d t_i^{\beta_i})\ (1 - \sum_{i=1}^d \log(t_i))^{1/2} . \tag{52}$$

Define

$$Lip(\boldsymbol{\beta},\frac{1}{2}) = \{f \in C(I^d) : \forall(\emptyset \neq A \subset \mathcal{D}) \;\; \omega_A(f,\boldsymbol{t}) = O(\omega_{\boldsymbol{\beta},\frac{1}{2}}(\boldsymbol{t};A))\} \tag{53}$$

and

$$lip(\boldsymbol{\beta},\frac{1}{2}) = \{f \in Lip(\boldsymbol{\beta},\frac{1}{2}) : \forall(\emptyset \neq A \subset \mathcal{D}) \;\; \omega_A(f,\boldsymbol{t}) = o(\omega_{\boldsymbol{\beta},\frac{1}{2}}(\boldsymbol{t};A))\} \tag{54}$$

where $O(\boldsymbol{t}(A))$ and $o(\boldsymbol{t}(A))$ refer to $\min(t_i : i \in A) \to 0$.

We will first prove the following proposition.

Proposition 7.

1. *We have* $Lip(\boldsymbol{\beta}) \subset Lip(\boldsymbol{\beta},\frac{1}{2})$ *and* $lip(\boldsymbol{\beta}) \subset lip(\boldsymbol{\beta},\frac{1}{2})$.
2. *If* $\boldsymbol{\beta}' < \boldsymbol{\beta}$ *then* $Lip(\boldsymbol{\beta}) \subset lip(\boldsymbol{\beta}')$.
3. *If* $\boldsymbol{\beta}' < \boldsymbol{\beta}$ *then* $Lip(\boldsymbol{\beta},\frac{1}{2}) \subset Lip(\boldsymbol{\beta}')$.

Proof of Proposition 7.

1. The first point is a consequence of $\omega_{\boldsymbol{\beta}}(\boldsymbol{t}) \leq \omega_{\boldsymbol{\beta},\frac{1}{2}}(\boldsymbol{t})$.
2. Let $f \in Lip(\boldsymbol{\beta})$ and $\boldsymbol{\beta}' < \boldsymbol{\beta}$. We know from (37) that $f \in Lip(\boldsymbol{\beta}')$. Let $\emptyset \neq A \subset \mathcal{D}$. Since $\boldsymbol{\beta}' < \boldsymbol{\beta}$ then

$$\frac{\omega_A(f,\boldsymbol{t})}{\omega_{\boldsymbol{\beta}'}(\boldsymbol{t};A)} \leq C\omega_{\boldsymbol{\beta}-\boldsymbol{\beta}'}(\boldsymbol{t};A) = o(\boldsymbol{t}(A)) \; .$$

   Hence $f \in lip(\boldsymbol{\beta})$.
3. Let $f \in Lip(\boldsymbol{\beta},\frac{1}{2})$ and $\boldsymbol{\beta}' < \boldsymbol{\beta}$. Let $\emptyset \neq A \subset \mathcal{D}$. Since $\boldsymbol{\beta}' < \boldsymbol{\beta}$ and $t\log t = o(1)$ when t goes to 0 then

$$\frac{\omega_A(f,\boldsymbol{t})}{\omega_{\boldsymbol{\beta}'}(\boldsymbol{t};A)} \leq C\omega_{\boldsymbol{\beta}-\boldsymbol{\beta}'}(\boldsymbol{t};A)(1 - \sum_{i\in A}\log(t_i))^{1/2} = o(\boldsymbol{t}(A)) \; .$$

   It follows that $f \in Lip(\boldsymbol{\beta}')$. □

In [25], Kamont proved that, with probability 1, the restrictions $B^{(\alpha)}_{I^d}$ of realizations of $B^{(\alpha)}$ to I^d satisfy

$$B^{(\alpha)}_{I^d} \in Lip(\alpha/2,\frac{1}{2}) \tag{55}$$

and

$$B^{(\alpha)}_{I^d} \notin lip(\alpha/2,\frac{1}{2}) \; . \tag{56}$$

We will prove the following result.

Proposition 8. *With probability 1, the restrictions* $B^{(\alpha)}_{I^d}$ *satisfy*

$$\forall\, i \in \mathcal{D} \qquad \alpha(I^d, e_i) = \alpha_i/2 \tag{57}$$

and

$$\forall y \in I^d \quad \forall\, i \in \mathcal{D} \qquad \alpha_l(y, e_i) = \alpha_i/2 \; . \tag{58}$$

Proof of Proposition 8. Using the third point in Proposition 7, relation (55) implies that, with probability 1, the restrictions $B^{(\alpha)}_{I^d}$ satisfy

$$B^{(\alpha)}_{I^d} \in Lip(\boldsymbol{\beta}) \qquad \forall \boldsymbol{\beta} < \alpha/2 \tag{59}$$

Using the second point in Proposition 7, relation (56) implies that, with probability 1, the restrictions $B^{(\alpha)}_{I^d}$ satisfy

$$B^{(\alpha)}_{I^d} \notin Lip(\boldsymbol{\beta}) \qquad \forall \boldsymbol{\beta} > \alpha/2\,. \tag{60}$$

Thanks to the first point in Proposition 3, relation (59) yields the lower bound in (57). The optimality of this lower bound cannot be deduced from (60). Nevertheless, the coefficients of $B^{(\alpha)}_{I^d}$ in the tensor product Schauder basis were obtained in [25]; in fact

$$B^{(\alpha)}_{I^d} = \sum_{\boldsymbol{j}\in M^d} \sum_{\boldsymbol{k}\in \tilde{N}_{\boldsymbol{j}}} C_{\boldsymbol{k}} \Phi_{\boldsymbol{k}}\ , \tag{61}$$

where $(C_{\boldsymbol{k}})_{\boldsymbol{k}\geq \boldsymbol{0}}$ is a Gaussian sequence, with $EC_{\boldsymbol{k}} = 0$, and the variance given by the formula

$$E|C_{\boldsymbol{k}}|^2 = \prod_{i=1}^{d} a_{k_i} \tag{62}$$

where

$$a_0 = 0,\ a_1 = 1 \text{ and } a_{k_i} = (2^{-\alpha_i} - 2^{-2})2^{-j_i\alpha_i} \ \text{ for } \ k_i \in \tilde{N}_{j_i}\ j_i \geq 0\,. \tag{63}$$

And clearly, the above optimality follows immediately from Theorem 1 (note that $B^{(\alpha)}_{I^d}$ is uniform Lipschitz on I^d since $B^{(\alpha)}_{I^d} \in C^{\tau}(I^d)$ for all $2\tau < min(\alpha_1, \cdots, \alpha_d)$).

The same arguments applied to the dilated and shifted field $\{\rho^{|\alpha|}B^{(\alpha)}(\rho^2 t - c) : t \in I^d\}$, $(\rho > 0,\ c \in \mathbb{R}^d)$ give the same result as in (57) if I^d is replaced by any arbitrary cube $Q \subset \mathbb{R}^d$. This implies that for any subset Ω of I^d with non-empty interior we have $\alpha(\Omega, e_i) = \alpha(I^d, e_i)$ for all $i \in \mathcal{D}$. Hence the local result (58). □

4.2. The Sierpinski cascade function

Without any loss of generality, we take $d = 2$. We will apply the above result to obtain both uniform and local critical directional Lipschitz regularity of the Sierpinski cascade function. It is a selfsimilar function adapted to a subdivision A used for the construction of Sierpinski carpet K (see [7,26,31] and references therein). In [7], we proved that the conjectures of Frisch and Parisi [17], or alternatively of Arneodo, Bacry and Muzy [4], called the multifractal formalism for functions, may fail for such selfsimilar functions. Our functions can modelize turbulence or cascade models; let s and t be two integers. We choose a finite subset A of $\{0, 1, \ldots, s-1\} \times \{0, 1, \ldots, t-1\}$ and for each pair $\omega = (i, j) \in A$, we consider the affine map $S_\omega : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$S_\omega(x_1, x_2) = \left(\frac{x_1}{s} + \frac{i}{s}, \frac{x_2}{t} + \frac{j}{t}\right)\ .$$

We construct an anisotropic Sierpinski carpet K as follows: we divide the unit square $\mathfrak{R} = I^2$ into a uniform grid of rectangles of height $1/t$ and width $1/s$; each S_ω maps the unit square into the rectangle

$$\mathfrak{R}_\omega = [\frac{i}{s}, \frac{i+1}{s}] \times [\frac{j}{t}, \frac{j+1}{t}].$$

The Sierpinski carpet K will be the unique non-empty compact set (see [19]) such that

$$K = \bigcup_{\omega \in A} S_\omega(K). \tag{64}$$

We have

$$K = \{x \in \mathfrak{R} \ : \ (S_{\omega_1} \circ \cdots \circ S_{\omega_n})^{-1}(x) \in \bigcup_{\omega \in A} \mathfrak{R}_\omega \quad \forall (\omega_1, \ldots, \omega_n) \in A^n\}$$
$$= \bigcap_{n=1}^{\infty} (\bigcup_{\omega \in A^n} \mathfrak{R}_\omega)$$

where

$$\mathfrak{R}_\omega = (S_{\omega_1} \circ \cdots \circ S_{\omega_n})(\mathfrak{R}) \quad \text{for} \quad \omega = (\omega_1, \ldots, \omega_n)\ .$$

Take $g(x) = \Lambda(x_1)\Lambda(x_2)$ with $\Lambda(t) = min(t, 1-t)$ if $t \in [0, 1]$ and 0 else. Clearly $\Lambda(t) = \frac{1}{2}\Phi_2(t)$.
We will call a the Sierpinski cascade function adapted to the subdivision A, a function F satisfying:

$$\forall\, x \in \mathfrak{R} \qquad F(x) = \sum_{\omega \in A} \lambda_\omega F(S_\omega^{-1}(x)) + g(x). \tag{65}$$

Iterating (65), we obtain for any $N \geq 2$:

$$\begin{aligned} F(x) \ = \ & g(x) + \sum_{n=1}^{N-1} \sum_{(\omega_1,\ldots,\omega_n)\in A^n} \lambda_{\omega_1} \cdots \lambda_{\omega_n} \ g\left(S_{\omega_n}^{-1} \cdots S_{\omega_1}^{-1}(x)\right) \\ & + \sum_{(\omega_1,\ldots,\omega_N)\in A^N} \lambda_{\omega_1} \cdots \lambda_{\omega_N} \ F\left(S_{\omega_N}^{-1} \cdots S_{\omega_1}^{-1}(x)\right). \end{aligned} \tag{66}$$

Define

$$|\lambda|_{max} = \max_{\omega \in A} |\lambda_\omega| \quad , \quad |\lambda|_{min} = \min_{\omega \in A} |\lambda_\omega| \quad \text{and} \quad \alpha_{min} = -\frac{\log |\lambda|_{max}}{\log \max\{s,t\}}\ .$$

The following result was obtained in [7].

Proposition 9. *If* $\sum_{\omega \in A} \mid \lambda_\omega \mid < st$, *then (65) has a unique solution in* $L^1(\mathfrak{R})$ *given by the series*

$$F(x) = g(x) + \sum_{n=1}^{\infty} \sum_{(\omega_1,\ldots,\omega_n)\in A^n} \lambda_{\omega_1} \cdots \lambda_{\omega_n} \ g\left(S_{\omega_n}^{-1} \cdots S_{\omega_1}^{-1}(x)\right). \tag{67}$$

If furthermore $\frac{1}{\max\{s,t\}} < |\lambda|_{max} < 1$, *then* $F \in C^{\alpha_{min}}(\mathfrak{R})$ *with* $0 < \alpha_{min} < 1$.

We will prove the following result.

Proposition 10. *Let S and T be two positive integers. Assume that $s = 2^S$ and $t = 2^T$. Assume that $\frac{1}{\max\{s,t\}} < |\lambda|_{max} < 1$. Then*

$$\alpha(\mathfrak{R}, e_1) = -\frac{\log|\lambda|_{max}}{\log s} \quad \textit{and} \quad \alpha(\mathfrak{R}, e_2) = -\frac{\log|\lambda|_{max}}{\log t} . \tag{68}$$

- *If $s < t$ then*

$$\forall\, e \neq e_1 \qquad \alpha(\mathfrak{R}, e) = \alpha_{min} = -\frac{\log|\lambda|_{max}}{\log t} . \tag{69}$$

- *If $s > t$ then*

$$\forall\, e \neq e_2 \qquad \alpha(\mathfrak{R}, e) = \alpha_{min} = -\frac{\log|\lambda|_{max}}{\log s} . \tag{70}$$

- *If $s = t$ then*

$$\forall\, e \qquad \alpha(\mathfrak{R}, e) = \alpha_{min} . \tag{71}$$

For any open set Ω of $\mathbb{R}^2$ with non-empty intersection with the Sierpinski carpet K, we have $\alpha(\Omega, e) = \alpha(\mathfrak{R}, e)$. This implies that the critical local directional Lipschitz regularity at any point y in K is constant $\alpha(\mathfrak{R}, e)$. But, if $y \notin K$, then it equals 1.

Proof of Proposition 10. Clearly if $\omega_l = (i_l, j_l)$ then

$$g\left(S^{-1}_{\omega_n} \cdots S^{-1}_{\omega_1}(x)\right) = \Lambda(s^n x_1 - s^{n-1} i_1 - \cdots - s i_{n-1} - i_n)\, \Lambda(t^n x_2 - t^{n-1} j_1 - \cdots - t j_{n-1} - j_n) .$$

Using Theorem 1, one gets immediately (68). Remark 6 yields (69) and (70). Result (71) is a consequence of Remark 6 and the fact that the maps S_ω are similitudes.

The local result in Proposition 10 follows from the selfsimilarity for $y \in K$, and the fact that if $y \notin K$ then F is a finite linear combination of translated dilated Λ function. □

5. Criteria of pointwise directional Lipschitz regularity

5.1. By decay conditions on two progressive difference in the given direction

It is known (see [18,20,21] and Remark 1) that, if a 1-parameter function f is uniformly Lipschitz regular on I, then its critical pointwise Lipschitz regularity can be characterized by estimates on the size of the Schauder coefficients;

Proposition 11. *If a 1-parameter function $f(x_1) = \sum_{j_1 \in M} \sum_{k_1 \in \tilde{N}_{j_1}} c_{k_1}(f) \phi_{k_1}(x_1)$ is uniformly Lipschitz regular on I, then for all $y_1 \in I$*

$$\alpha_p(y_1) = \min\left(1, \liminf_{j_1 \to \infty} \inf_{k_1 \in \tilde{N}_{j_1}} \frac{\log(|c_{k_1}(f)|)}{\log(2^{-j_1} + |n_1 2^{-j_1} - y_1|)}\right) . \tag{72}$$

Let now $d \geq 2$, $f \in C(I^d)$ and $y = (y_1, \ldots, y_d) \in I^d$. Clearly,

$$f(x_1, y_2, \ldots, y_d) = \sum_{j_1 \in M} \sum_{k_1 \in \tilde{N}_{j_1}} c_{k_1}(f(., y_2, \ldots, y_d))\, \phi_{k_1}(x_1)\,, \tag{73}$$

with

$$c_0(f(., y_2, \ldots, y_d)) = f(0, y_2, \ldots, y_d)\,, \tag{74}$$

$$c_1(f(., y_2, \ldots, y_d)) = f(1, y_2, \ldots, y_d) - f(0, y_2, \ldots, y_d) \tag{75}$$

and for $k_1 = 2^{j_1} + n_1 \in \tilde{N}_{j_1}$ with $j_1 \geq 0$

$$c_{k_1}(f(., y_2, \ldots, y_d)) = f(\frac{2n_1 - 1}{2^{j_1+1}}, y_2, \ldots, y_d) - \frac{1}{2}(f(\frac{n_1 - 1}{2^{j_1}}, y_2, \ldots, y_d) + f(\frac{n_1}{2^{j_1}}, y_2, \ldots, y_d))\,. \tag{76}$$

Put

$$\beta(y, e_1) = \liminf_{j_1 \to \infty} \inf_{k_1 \in \tilde{N}_{j_1}} \frac{\log |c_{k_1}(f(., y_2, \ldots, y_d))|}{\log (2^{-j_1} + |n_1 2^{-j_1} - y_1|)}\,. \tag{77}$$

The following result is a criterium of pointwise directional Lipschitz regularity by decay conditions on two progressive difference in the given direction.

Theorem 2. *Let $d \geq 2$, $y \in I^d$ and $f \in C(I^d)$. If f is uniformly Lipschitz regular on I^d in direction e_1 then the critical pointwise Lipschitz regularity of f at $y \in I^d$ in direction e_1 is given by*

$$\alpha_p(y, e_1) = \min (1, \beta(y, e_1))\,. \tag{78}$$

Remark 8. The previous result remains valid if we replace the coefficients $c_{k_1}(f(., y_2, \ldots, y_d))$ by those of the expansion of the function in a (smooth) wavelet basis (i.e., $\tau > 1$).

5.2. By decay conditions for the coefficients of the expansion of f in a tensor product Schauder basis

We will now express critical pointwise Lipschitz regularity in terms of decay conditions for the coefficients $C_{\boldsymbol{k}}(f)$ (given in (44)) of f in a tensor product Schauder basis.

By comparing (73) to expansion (45), we get

$$c_{k_1}(f(., y_2, \ldots, y_d)) = \sum_{(j_2, \ldots, j_d) \in M^{d-1}} \sum_{(k_2, \ldots, k_d) \in \prod_{i=2}^{d} \tilde{N}_{j_i}} C_{\boldsymbol{k}}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i)\,, \tag{79}$$

where $\boldsymbol{k} = (k_1, k_2, \ldots, k_d)$.

Put

$$\gamma(y, e_1) = \liminf_{j_1 \to \infty} \inf_{k_1 \in \tilde{N}_{j_1}} \frac{\log \left(\left| \sum_{(j_2, \ldots, j_d) \in M^{d-1}} \sum_{(k_2, \ldots, k_d) \in \prod_{i=2}^{d} \tilde{N}_{j_i}} C_{\boldsymbol{k}}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) \right| \right)}{\log (2^{-j_1} + |n_1 2^{-j_1} - y_1|)}\,. \tag{80}$$

Clearly

$$\beta(y, e_1) = \gamma(y, e_1)\,.$$

Theorem 3. *Let $d \geq 2$, $y \in I^d$ and $f \in C(I^d)$. If f is uniformly Lipschitz regular on I^d in direction e_1 then the critical pointwise Lipschitz regularity of f at $y \in I^d$ in direction e_1 is given by*

$$\alpha_p(y, e_1) = \min\left(1, \gamma(y, e_1)\right) \ . \tag{81}$$

Remark 9. The previous result remains valid if we replace the coefficients $C_{\boldsymbol{k}}(f)$ by those of the expansion of the function in the basis of hyperbolic tensor products of smooth wavelet functions [16]. Of course, Schauder terms $\phi_{k_i}(y_i)$ are replaced by wavelet terms $\psi_{k_i}(y_i)$.

If $k_i \in \tilde{N}_{j_i}$, with $k_i \geq 2$ then ϕ_{k_i} has support $[(n_i - 1)2^{-j_i}, n_i 2^{-j_i}]$. It follows that for $j_i \in M$ with $j_i \geq 0$ and $y \in I^d$, there exists a unique value of $k_i = k_{i,y}$ for which $y_i \in [(n_i - 1)2^{-j_i}, n_i 2^{-j_i})$. We keep the notation $k_i = k_{i,y}$ even if $j_i \in \{-2, -1\}$. Put

$$\boldsymbol{k}(y) = (k_1, \ldots, k_d) \in \tilde{N}_{\boldsymbol{j}} \ .$$

It follows that

$$\gamma(y, e_1) = \liminf_{j_1 \to \infty} \frac{\log\left(\left|\displaystyle\sum_{(j_2,\ldots,j_d)\in M^{d-1}} C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i)\right|\right)}{\log\left(2^{-j_1} + |n_1 2^{-j_1} - y_1|\right)} \ . \tag{82}$$

Corollary 1. *Let $d \geq 2$, $y \in I^d$ and $f \in C(I^d)$. If f is uniformly Lipschitz regular on I^d in direction e_1 then the critical pointwise Lipschitz regularity of f at $y \in I^d$ in direction e_1 is given by*

$$\alpha_p(y, e_1) = \min\left(1, \gamma(y, e_1)\right) \ . \tag{83}$$

Remark 10. If hyperbolic tensor products of Schauder functions are replaced by hyperbolic tensor products of smooth compactly supported wavelet functions, then the previous corollary remains valid if in (80) we keep only values of $k_i = k_{i,y}$ for which y_i is inside the support of the wavelet functions.

Remark 11. Summation over $(j_2, \ldots, j_d) \in M^{d-1}$ in (82) makes the use of Corollary 1 a little bit difficult. For this reason, we aim to avoid that summation. We will see that the positivity of Schauder functions ϕ_{k_i} is important. Of course this is not the case for general (smooth compactly supported) wavelet functions.

Put

$$\rho(y, e_1) = \liminf_{j_1 \to \infty} \inf_{(j_2,\ldots,j_d)\in M^{d-1}} \frac{\log\left(|C_{\boldsymbol{k}(y)}(f)| \displaystyle\prod_{i=2}^{d} \phi_{k_i}(y_i)\right)}{\log\left(2^{-j_1} + |n_1 2^{-j_1} - y_1|\right)} \ . \tag{84}$$

Theorem 4. *Let f be uniformly Lipschitz regular on I^d. Let $y \in I^d$.*

If

$$\forall\, \boldsymbol{k} \quad C_{\boldsymbol{k}(y)}(f) \geq 0 \tag{85}$$

then

$$\alpha_p(y, e_1) = \min\left(1, \rho(y, e_1)\right) \ . \tag{86}$$

Proof of Theorem 4. Thanks to Corollary 1, it suffices to prove equality between $\rho(y, e_1)$ and $\beta(y, e_1)$.

By assumption (85)

$$C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) \leq \sum_{(j_2,\ldots,j_d)\in M^{d-1}} C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) \; .$$

Since functions ϕ_{k_i} are positive then it follows that

$$\rho(y, e_1) \geq \beta(y, e_1) \; . \tag{87}$$

On the other hand, since f is uniformly Lipschitz regular on I^d, then using the second result in Proposition 3, there exists $\delta > 0$ such that

$$\forall\, \mathbf{0} < \boldsymbol{\theta} = (\theta_1, \cdots, \theta_d) \text{ with } |\boldsymbol{\theta}| \leq 1 \quad f \in Lip(\theta_1\delta, \cdots, \theta_d\delta) \; . \tag{88}$$

Therefore from Proposition 5, there exists $C > 0$ such that

$$\forall\, \boldsymbol{k} \quad |C_{\boldsymbol{k}}(f)| \leq C \prod_{i=1}^{d} 2^{-j_i\theta_i\delta} \; . \tag{89}$$

Let $0 < \sigma < 1$. We will again use the positivity of functions ϕ_{k_i}.

Write

$$\begin{aligned} C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) &= \left(C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) \right)^{\sigma} \left(C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) \right)^{1-\sigma} \\ &\leq \left(C_{\boldsymbol{k}(y)}(f) \right)^{\sigma} \left(C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) \right)^{1-\sigma} \; . \end{aligned}$$

By (89)

$$C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) \leq C \left(\prod_{i=1}^{d} 2^{-j_i\theta_i\delta\sigma} \right) \left(C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) \right)^{1-\sigma} \; .$$

Using the convergence of series $\sum_{j_i} 2^{-j_i\theta_i\delta\sigma}$, we deduce that

$$\sum_{(j_2,\ldots,j_d)\in M^{d-1}} C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) \leq C \left(C_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i) \right)^{1-\sigma} \; .$$

Consequently

$$\beta(y, e_1) \geq (1-\sigma)\rho(y, e_1) \; . \tag{90}$$

Since (90) holds for any $\sigma \in (0, 1)$, we conclude that

$$\beta(y, e_1) \geq \rho(y, e_1) \; . \quad \square \tag{91}$$

Corollary 2. *Let f be uniformly Lipschitz regular on I^d. We put*

$$C_K^+(f) = \max\{C_K(f), 0\} \quad \text{and} \quad C_K^-(f) = -\min\{C_K(f), 0\} \ . \tag{92}$$

Let $y \in I^d$. We set

$$\rho^+(y, e_1) = \liminf_{j_1 \to \infty} \inf_{(j_2,\ldots,j_d) \in M^{d-1}} \frac{\log\left(C^+_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i)\right)}{\log\left(2^{-j_1} + |n_1 2^{-j_1} - y_1|\right)} \tag{93}$$

and

$$\rho^-(y, e_1) = \liminf_{j_1 \to \infty} \inf_{(j_2,\ldots,j_d) \in M^{d-1}} \frac{\log\left(C^-_{\boldsymbol{k}(y)}(f) \prod_{i=2}^{d} \phi_{k_i}(y_i)\right)}{\log\left(2^{-j_1} + |n_1 2^{-j_1} - y_1|\right)} \ . \tag{94}$$

1. *If* $\min(1, \rho^+(y, e_1), \rho^-(y, e_1)) = 1$ *then* $\alpha_p(y, e_1) = 1$.
2. *If* $\min(1, \rho^+(y, e_1), \rho^-(y, e_1)) < 1$ *then*
   (a) *If* $\rho^+(y, e_1) \neq \rho^-(y, e_1)$ *then* $\alpha_p(y, e_1) = \min(\rho^+(y, e_1), \rho^-(y, e_1))$.
   (b) *If* $\rho^+(y, e_1) = \rho^-(y, e_1)$ *then* $\alpha_p(y, e_1) \geq \rho^+(y, e_1)$.

Proof of Corollary 2. It suffices to split f as

$$f = \sum_{\boldsymbol{j} \in M^d} \sum_{\boldsymbol{k} \in \tilde{N}_{\boldsymbol{j}}} C^+_{\boldsymbol{k}}(f) \Phi_{\boldsymbol{k}} \ - \ \sum_{\boldsymbol{j} \in M^d} \sum_{\boldsymbol{k} \in \tilde{N}_{\boldsymbol{j}}} C^-_{\boldsymbol{k}}(f) \Phi_{\boldsymbol{k}} \tag{95}$$

and apply Theorem 4 for both left and right hand term functions in (95). □

Acknowledgments

Mourad Ben Slimane and Borhen Halouani extend their appreciation to the Deanship of Scientific Research at King Saud University for funding this work through research group No. (RG-1435-063).

Mourad Ben Slimane would like to thank Stéphane Jaffard for stimulating discussions. The authors are grateful the referee(s) for his/her (their) suggestions that have led to section 5.

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