# 5- Fubini's Theorem

## 5.1 Product Measure Spaces:

Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ 2 measure spaces. We intend to construct the product measure on a suitable $\sigma$-algebra contained in the power set of the Cartesian product $X = X_1 \times X_2$. By a rectangular set $R$ in $X$, we mean any set of the form $R = A \times B$ where $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$. We will take as the family of elementary sets for the product measure

$$\mathcal{E} = \{ E = \bigcup_{j=1}^{n} R_j \; ; \; R_j = A_j \times B_j \; ; \; A_j \in \mathcal{A}_1, \; B_j \times \mathcal{A}_2 \} \; (*)$$

where $R_j$ are disjoint rectangles and $n$ is an arbitrary natural number. $\mathcal{E}$ is an <u>algebra</u>.

<u>Def</u> We define the product measure $\mu_1 \otimes \mu_2 (E) = \sum_{j=1}^{n} \mu_1(A_j)\mu_2(B_j)$

for each elementary set $E \in \mathcal{E}$ as defined by $(*)$.

⚠ This definition requires justification because the decomposition given in equation $(*)$ is <u>not</u> unique.

$$E = \bigcup_{j=1}^{n} (A_j \times B_j) = \bigcup_{j=1}^{m} (C_j \times D_j)$$

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It follows from the finite additivity of each of the measures $\mu_1$ and $\mu_2$ that:

$$\mu_1(A_j)\,\mu_2(B_j) = \sum_{k=1}^{m} \mu_1(A_j \cap C_k)\,\mu_2(B_j \cap D_k)$$

and

$$\mu_1(C_k)\,\mu_2(D_k) = \sum_{j=1}^{n} \mu_1(A_j \cap C_k)\,\mu_2(B_j \cap D_k)$$

$$\Rightarrow \sum_{k=1}^{m} \mu_1(C_k)\,\mu_2(D_k) = \sum_{j=1}^{n} \mu_1(A_j)\,\mu_2(B_j) \;\blacksquare$$

<u>Lemma 5.2</u>: Suppose $A \times B = \bigcup_{j=1}^{\infty}(A_j \times B_j)$ where $A, A_j \in \mathcal{A}_1$ and $B, B_j \in \mathcal{A}_2$ and the $(A_j \times B_j)$ are <u>disjoint</u>. Then $\mu_1 \otimes \mu_2 (A \times B) = \sum_{j=1}^{\infty} \mu_1 \otimes \mu_2 (A_j \times B_j)$.

<u>Proof</u>: we have: $\chi_A(x)\,\chi_B(y) = \chi_{A \times B}(x,y) = \sum_{j=1}^{\infty} \chi_{A_j}(x)\,\chi_{B_j}(y)$

By the Monotone Convergence Theorem

$$\chi_A(x)\,\mu_2(B) = \sum_{j=1}^{\infty} \chi_{A_j}(x)\,\mu_2(B_j),$$

and also by the Monotone convergence Th.m:

$$\mu(A \times B) = \mu_1(A)\,\mu_2(B) = \sum_{j=1}^{\infty} \mu(A_j)\,\mu_2(B_j) \;\blacksquare$$

**Def:** If $E \subset X_1 \times X_2$ ; we define the x-section of E by
$$E_x = \{ y \in X_2 , (x,y) \in E \} ; y \in X_2$$ and
the y-section by
$$E^y = \{ x \in X_1 , (x,y) \in E \} ; y \in X_2 .$$
Similarly if $f : X \longrightarrow \overline{\mathbb{R}}$, then the x and y-sections of f are the mappings $f_x : X_2 \longrightarrow \overline{\mathbb{R}}$ and $f^y : X_1 \longrightarrow \overline{\mathbb{R}}$ defined by $f_x(y) = f(x,y)$ and $f^y(x) = f(x,y)$.

## 5.2 Fubini - Tonelli's Theorem:

**Theorem** (Fubini Tonelli)

Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be 2 <u>σ-finite</u> measure spaces, and let the product measure space be denoted by $(X, \mathcal{A}, \mu)$. Let f be a <u>non negative</u> measurable function on X. Then the functions: $g(x) = \int_{X_2} f(x,y)\, d\mu_2(y)$ and $R(y) = \int_{X_1} f(x,y)\, d\mu_1(x)$ are measurable on $X_1$ and $X_2$ respectively and:
$$\iint_X f(x,y)\, d\mu(x,y) = \int_{X_2} \left( \int_{X_1} f(x,y) d\mu_1(x) \right) d\mu_2(y)$$
$$= \int_{X_1} \left( \int_{X_2} f(x,y) d\mu_2(y) \right) d\mu_1(x)$$

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These 3 integrals may be $+\infty$.

**Fubini's Thm:**

Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be 2 <u>σ-finite</u> measure spaces, and ~~let~~ the product measure space be denoted by $(X, \mathcal{A}, \mu)$. Let $f \in L^1(X, d\mu)$. Then the functions $\int_{X_2} f(x,y)\, d\mu_2(y) \in L^1(X_1, \mu_1)$ and $\int_{X_1} f(x,y) d\mu_1(x) \in L^1(X_2, \mu_2)$ and:
$$\iint_X f(x,y)\, d\mu(x,y) = \int_{X_1} \left( \int_{X_2} f(x,y) d\mu_2(y) \right) d\mu_1(x)$$
$$= \int_{X_2} \left( \int_{X_1} f(x,y) d\mu_1(x) \right) d\mu_2(x)$$
holds.

**Example:**
$$f(x,y) = \begin{cases} \dfrac{x-y}{(x+y)^3} & \text{if } x,y \in (0,1] \\ 0 & \text{otherwise.} \end{cases}$$
$$\int dx \left( \int f(x,y) dy \right) = \int_0^1 dx \int_0^1 \left( \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right) dy = \int_0^1 \frac{dx}{(1+x)^2}$$
$$= 1/2 .$$

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but $\int dy \left(\int f(x,y)\, dx\right) = -1/2$.

( Here $f$ is a borelian function but $f$ is not lebesgue integrable).

* The strategy of the proof of Fubini's Thm is to begin by proving the result for characteristic functions of rectangles, then simple functions and extend the result to general measurable functions on $X$.

**Proposition 5.6** If $E \in \mathcal{A}$ then the sections $E_x$ and $E^y$ resp. belong to $\mathcal{A}_2$ for each $x \in X_1$, and to $\mathcal{A}_1$ for each $y \in X_2$. If $f$ is measurable with respect to the product algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ then its sections $f_x$ and $f^y$ are measurable with respect to the factors $\mathcal{A}_2$ and $\mathcal{A}_1$ resp.

**Proof**: Let $\mathcal{C}$ be the collection of all subsets $E \subset X$ such that $E_x \in \mathcal{A}_2$ for all $x \in X_1$ and $E^y \in \mathcal{A}_1$ for all $y \in X_2$. Then $(A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \phi & \text{if } x \in A^c \end{cases}$ and similarly for the section $(A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \phi & \text{if } y \in B^c \end{cases}$

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Hence $\mathcal{C}$ contains all rectangles. Moreover $\mathcal{C}$ is $\sigma$-algebra, since $\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} (E_j)_x$ and $(E_x)^c = (E^c)_x$ and similarly for $y$-sections. Therefore $\mathcal{C} \subset \mathcal{A}_1 \otimes \mathcal{A}_2$.

The measurability of $f_x$ and $f^y$ follows from the first statement and the relationships:

$$(f_x)^{-1}(B) = (f^{-1}(B))_x \quad ; \quad (f^y)^{-1}(B) = (f^{-1}(B))^y \blacksquare$$

**Lemma 5.7**:

Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be two $\sigma$-finite measure spaces and $(X, \mathcal{A}, \mu)$ be the product measure space. Given $E \in \mathcal{A}$, the sections $(\chi_E)_x$ and $(\chi_E)^y$ are measurable in $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ respectively; and

(*)
$$\mu(E) = \iint_X \chi_E(x,y)\, d\mu(x,y) = \int_{X_2} \left( \int_{X_1} (\chi_E)^y(x)\, d\mu_1(x) \right) d\mu_2$$
$$= \int_{X_1} \left( \int_{X_2} (\chi_E)_x(y)\, d\mu_2(y) \right) d\mu_1(x).$$

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**Proof**: We shall establish the lemma for the case in which $\mu_1$ and $\mu_2$ are finite measures. Let $\mathscr{C}$ be the class of sets in $\mathscr{A}$ for which the lemma holds. When $E$ is a rectangle, $E = A\times B$, $(\chi_E)^y(x) = (\chi_E)_x(y) = \chi_A(x)\,\chi_B(y)$ and (*) is equal to $\mu_1(A)\cdot\mu_2(B) = \mu(E)$. Then $E\in\mathscr{C}$.

. $\mathscr{C}$ contains finite disjoint rectangles. It suffices to prove that $\mathscr{C}$ is a monotone class.

If $E = \bigcup_{j=1}^{\infty} E_j$ with $(E_j)\nearrow$ of sets of $\mathscr{C}$. Then since $\mu_1$ and $\mu_2$ are finite measures then by the monotone Convergence Thm $E\in\mathscr{C}$.

Proof (Tonelli's Thm)

This lemma above proves that thm is valid for charachteristic functions for measurable subsets and by additivity the thm is valid for simples functions. If $f$ is non negative measurable on $(X,\mathscr{A},\mu)$ there exist an increasing sequence of simples functions and the result is deduced from Monotone Convergence Thm.

Proof (Fubini' Thm)

If $f$ is integrable on $X$, we decompose $f = f^+ - f^-$ and we apply Tonelli's Thm for $f^+$ and $f^-$.

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**example**: Let $g(x) = \begin{cases} 1/\sqrt{x} & \text{if } 0 < x \le 1. \\ \frac{1}{x^2} & \text{if } x > 1. \end{cases}$ $\quad g(x) > 0$ on $(0,\infty)$

$$\int_0^\infty g(x)\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx + \int_1^\infty \frac{dx}{x^2} = 3$$

Now take $f(x,y) = \begin{cases} g(x-y) & \text{if } x>y \\ 0 & \text{if } x=y \\ -g(y-x) & \text{if } x<y \end{cases}$

$$\int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^\infty g(x-y)\,dx + \int_{-\infty}^{y} -g(y-x)\,dx$$
$$= \int_0^\infty g(u)\,du + \int_\infty^0 g(u)\,du = 0$$

Similarly $\int f(x,y)\,dy = 0$

But $\int f(x,y)\,dx\,dy$ does not exist. It means $f$ is not Lebesgue integrable.

because: $f = f^+ - f^-$ ; $\quad f^+(x,y) = g(x-y)\,\chi_{\{x>y\}}(x,y)$.

$f \notin L^1([0,\infty]^2, d\mu)$ $\quad \int f^+\,dx\,dy = \int_{\mathbb{R}} dy \Big( \underbrace{\int_y^\infty g(x-y)\,dx}_{=3} \Big) = \infty.$

Prop 5.6 ① Every section of a measurable set is a measurable set.
② Every section of a measurable function is a measurable function.

Proof: ① Let $\mathcal{E}$ be the class of all those subsets of $X_1 \times X_2$ which have the property that each of their sections is measurable. If $E = A \times B$ is a measurable rectangle, then every section of $E$ is either empty or else equal to one of the sides, ($A$ or $B$ according as the section is a $X_2$-section or an $X_1$-section) and therefore $E \in \mathcal{E}$. Since it is easy to verify that $\mathcal{E}$ is a σ-algebra it follows $S \times T \subset \mathcal{E}$.

② If $f$ is a measurable function on $X_1 \times X_2$, if $x \in X_1$ and if $B$ is any Borel set on the real line, then the measurability of $f_x^{-1}(B)$ follows from ① and the relations:

$$f_x^{-1}(B) = \{y \in X_2 \ / \ f_x(y) \in B\} = \{y \mid f(x,y) \in B\}$$
$$= \{y \mid (x,y) \in f^{-1}(B)\} = \left(f^{-1}(B)\right)_x.$$

The proof of the measurability of an arbitrary $X_2$-section of $f$ is similar ∎

⚠ - For ②, conversely it is false.
Take $X_1 = X_2 = \mathbb{R}$ and $\mathcal{E}$ the σ-algebra generated by $\{x\}$.
$f = \chi_\Delta$ where $\Delta = \{(x,x) \ / \ x \in \mathbb{R}\}$ (the diagonal).
$f$ is not measurable function for $\mathcal{E} \otimes \mathcal{E}$. But $f_x$ and $f^y$ are measurable functions.