examples: ① Let g is a Borelian bounded on $\mathbb{R}_+$.

$h(t) = \int_0^\infty e^{-tx} g(x)\,dx$ is defined and infinitely differentiate on $(0,\infty)$

Precisely $f(t,x) = e^{-tx} g(x) \chi_{[0,\infty)}(x)$

$$\frac{\partial^n}{\partial t^n} f(t,x) = (-x)^n e^{-tx} g(x) \chi_{[0,\infty)}(x)$$

$\forall n \in \mathbb{N}$, $\left|\frac{\partial^n}{\partial t^n} f(t,x)\right| \leq g_n(x) \quad \forall t \in I = [0,\infty)$

$g_n(x) = \alpha_n e^{-ax} \chi_{[0,\infty)}$ are integrable on $\mathbb{R}$ by the Lebesgue measure.

We prove by induction on n:

h is n-derivable

$h^{(n)}(t) = \int_0^\infty (-x)^n e^{-tx} g(x)\,dx.$

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4 - Comparison of Riemann and Lebesgue Integrals

4.1 Riemann and Lebesgue Integrals

Let a and b 2 reals numbers, $a<b$. We consider the measure space $([a,b], \mathcal{B}^*, \lambda)$ where λ is the Lebesgue measure on $\mathbb{R}$ and $\mathcal{B}^*$ is the Lebesgue σ-algebra of $[a,b]$. For a bounded function f on $[a,b]$, we denote $\int_a^b f(x)dx$ the Riemann integral for f on $[a,b]$ and $\int_a^b f(x)\,d\lambda(x)$ the Lebesgue integral, if they exist.

- Let f be a bounded function on $[a,b]$. Then from the definition of the Riemann integral and the properties of the lower and upper Darboux sum of f, there exits an increasing sequence of partitions $(\sigma_n)_n$ of $[a,b]$ such that if $\sigma_n = \{x_0 = a, \dots, x_{p_n} = b\}$, the sequence $(\delta_n)_n$ defined by $\delta_n = \sup_{0\leq k\leq p_n-1} |x_{k+1} - x_k|$ converge to 0. (δ_n is called the norm of the partition). We denote:

$$(U) \quad \int_a^b f(x)dx = \lim_{n\to\infty} S(\sigma_n, f)$$

$$(L) \quad \int_a^b f(x)dx = \lim_{n\to\infty} s(\sigma_n, f)$$

Let $(g_n)_n$ and $(h_n)_n$ the sequences of simple functions defined by:

$$\boxed{\begin{aligned} g_n(x) &= \begin{cases} m_k = \inf_{[x_k, x_{k+1})} f(t) & \text{if } x_k \leq x < x_{k+1} \text{ and } g_n(b) = f(b) \end{cases} \\ h_n(x) &= \begin{cases} M_k = \sup_{[x_k, x_{k+1})} f(t) & \text{if } x_k \leq x < x_{k+1} \text{ and } h_n(b) = f(b) \end{cases} \end{aligned}}$$

- $(g_n) \nearrow$ and $(h_n) \searrow$ on $[a,b]$.

For $x \in [a,b]$, $\quad g_n \xrightarrow[n\to\infty]{} g$ and $h_n \xrightarrow[n\to\infty]{} h$

We remark that

(Upper Sum) $\quad S(\sigma_n, f) = \int_a^b h_n(x)\,dx = \int_a^b h_n(x)\,d\lambda(x)$

(Lower Sum) $\quad s(\sigma_n, f) = \int_a^b g_n(x)\,dx = \int_a^b g_n(x)\,d\lambda(x)$

Since g and h are measurables. It follows from the Monotone Lebesgue theorem

(I) $\quad \lim_{n\to\infty} \int_a^b g_n(x)\,dx = \int_a^b g(x)\,dx = \int_a^b g(x)\,d\lambda(x)$ (Lower sum)

(II) $\quad \lim_{n\to\infty} \int_a^b h_n(x)\,dx = \int_a^b h(x)\,dx = \int_a^b f(x)\,d\lambda(x)$ (Upper Sum)

in other hand we have $g(x) \le f(x) \le h(x) \quad \forall x \in [a,b]$

Thm Let f be a bounded function on $[a,b]$.

a) If f is Riemann-integrable on $[a,b]$, the f is Lebesgue integrable on $[a,b]$ and $\quad \int_a f(x)\,d\lambda(x) = \int_a^b f(x)\,dx$.

b) f is Riemann-integral on $[a,b]$ if and only if, the set of discontinuity of f is a null set.

c) If the set of discontinuity of f is a null set, then f is Lebesgue integrable and $\quad \int_a^b f(x)\,d\lambda(x) = \int_a^b f(x)\,dx$.

Proof: a) If f is Riemann integrable on $[a,b]$, we have:

$$(L) \int_a^b f(x)dx = (U)\int_a^b f(x)\,dx = \int_a^b f(x)dx$$

From (I) and (II), we shall have $\quad \int_{[a,b]} g(x)\,d\lambda(x) = \int_{[a,b]} h(x)\,d\lambda(x)$

Thus $\int_a^b [h(x) - g(x)]\,d\lambda(x) = 0 \Rightarrow (h-g)$ is a non negative integrable function then $h = g$ a.e and then $f = g$ a.e on $[a,b]$. Thus f is measurable

and $\int_a^b f(x)\,dx = \int_a^b f(x)\,d\lambda(x)$.

b) f is Riemann-integrable $\Leftrightarrow (U): \int_a^b f(x)\,dx = (L) \int_a^b f(x)dx \Rightarrow h = g$ a.e

Lemma:
Let f, g and h as above. For $x \in [a,b] \setminus (\bigcup_n \sigma_n)$; $g(x) = h(x) \Leftrightarrow f$ is Continuous in the point x.

We deduce from the above lemma; the fact:

f is Riemann-integrable $\Leftrightarrow h = g$ a.e which is equivalent to $\{x / h(x) \neq g(x)\} \cup (\bigcup_n \sigma_n)$ is a null set for the Lebesgue measure λ and f is continuous a.e on $[a,b]$.

this is equivalent that the set of discontinuity of f is a null set.

c) If the set of discontinuity of f is a null set. Then $\lim_n g_n = \lim_{n\to\infty} h_n = f$ at each point of continuity of f, then f is measurable and by Dominate convergence theorem yields

$$\lim_{n\to\infty} \int_a^b g_n(x)\,dx = \int_a^b f(x)\,dx \quad . \text{ Thus } f \text{ is Riemann integrable}$$

$$\lim_{n\to\infty} \int_a^b h_n(x)\,dx = \int_a^b f(x)\,dx$$

$$\text{and } \int_a^b f(x)\,dx = \int_a^b f(x)\,d\lambda(x). \quad \blacksquare$$

Prop Let $f : [a,b] \longrightarrow \mathbb{R}$ is a bounded function.
f is Riemann integrable $\Leftrightarrow$ f is continuous a.e on $[a,b]$

proof: a) Suppose f is Rieman integrable.
For $x \in [a,b]$, set $g(x) = \sup_{\delta>0} \inf_{\substack{|y-x|<\delta \\ y\in[a,b]}} f(y)$; $h(x) = \inf_{\delta>0} \sup_{\substack{|y-x|<\delta \\ y\in[a,b]}} f(y)$

So f is continuous at $x \Leftrightarrow g(x) = h(x)$.

we have $g \leq f \leq h$

because f is Rieman integrable then both g and h are Rieman Integrable.

$\int g(x)\,d\lambda(x) = \int h(x)\,d\lambda(x) = \int f(x)\,d\lambda(x)$

$\Rightarrow g = h$ a.e So f is continuous a.e on $[a,b]$.

(b) Now we suppose f is continuous a.e on $[a,b]$.
For each $n \in \mathbb{N}$, let σ_n be the partition of $[a,b]$ into 2^n equal intervals.
Set $h_n(x) = \sup_{y\in(c,d)} f(y)$; $g_n = \inf_{(c,d)} f(y)$

if (c,d) is open interval of σ_n containing x, we say $h_n(x) = g_n(x) = f(x)$

if x is one of the points of the list σ_n.

Then $(g_n) \nearrow$ and $(h_n) \searrow$ and each function on each of finite family of intervals covering $[a,b]$ and

$$s(f,\sigma_n) = \int g_n(x)\,d\mu(x)$$

$$S(f,\sigma_n) = \int h_n(x)\,d\mu(x)$$

Next, $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} h_n(x) = f(x)$ at any point x at which f is continuous.

So $f = \lim_n g_n = \lim_n h_n$ a.e.

By Lebesgue's Dominate Cvg, $\lim_n \int g_n\,d\mu = \int f\,d\mu = \lim_n \int h_n\,d\mu$

But It means $S(f) \leq \int f\,d\mu \leq s(f)$. So these are all equal and f is Rieman integral. $\blacksquare$

4.2 Generalized Integral and Lebesgue Integral:

Let (a,b) be an open integral of $\mathbb{R}$ and let f be a locally Riemann-integ

function on (a,b) (ie f is Riemann-integrable on $[\alpha,\beta]$ for all α,β / $a<\alpha<\beta<b$)
We say that the generalized Riemann integral of f exists (x_0 fixed in (a,b)).
This limit when it exists does not depend on x_0 and is denoted by $\int_a^b f(x)dx$.

Proposition: Let f be a locally Riemann-integrable function defined on (a,b). Then f is Lebesgue integrable on (a,b) if and only if the improper integral $\int_a^b f(x)dx$ is absolutely convergent and in this case the Riemann integral and the Lebesgue integral coincide
(ie $\int_a^b f(x)\,dx = \int_a^b f(x)\,d\lambda(x)$).

proof: we consider 2 sequences $(a_n)_n$ and $(b_n)_n$ of (a,b) such that $(a_n) \downarrow$ to a and $(b_n) \nearrow$ to b. Let $\varphi_n(x) = |f(x)|\,\chi_{[a_n,b_n]}$

- $(\varphi_n)\nearrow$, $\varphi_n \xrightarrow[n\to\infty]{} |f|\chi_{[a,b]}$; φ_n are measurable then f is measurable

It follows from Monotone Convergence thm that:

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n(x)\, d\lambda(x) = \int_a^b |f(x)|\, dx \quad . \quad (*)$$

Moreover f is Lebesgue integrable. To show that the two integrals coincide we set:

$g_n = f\chi_{[a_n,b_n]}$. Then $g_n \xrightarrow[n\to\infty]{} f\chi_{[a,b]}$. The functions g_n are integrable and $|g_n| \le |f|_{[a,b]}$. It follows by the Dominated Convergence thm that:

$$\lim_{n\to\infty} \int_a^b g_n(x)\, d\lambda(x) = \int_a^b f(x)\, d\lambda(x).$$

From $(*)$, we get $\int_a^b |f(x)|dx = \int_a^b |f(x)|\,d\lambda(x)$.

Conversely; If f is Lebesgue-integrable on (a,b), then $|f|$ is Lebesgue integrable on (a,b).

put $f_n = |f|\chi_{[a_n,b_n]}$. This sequence fulfill the hypotheses of the monotone cvg thm then $\lim_{n\to\infty} \int_a^b f_n(x)d\lambda(x) = \int_a^b f(x)d\lambda(x) < \infty$

Moreover $\int_a^b f_n(x)\, d\lambda(x) = \int_{a_n}^{b_n} |f(x)|dx$ which follows from the previous thm. Then

$$\lim_{n\to\infty} \int_{a_n}^{b_n} |f(x)|\,dx \text{ exists in } \mathbb{R}. \text{ Then } \int_a^b |f(x)|dx < \infty.$$

5- Fubini's Theorem

5.1 Product Measure Spaces:

Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ 2 measure spaces. We intend to construct the product measure on a suitable σ-algebra contained in the power set of the Cartesian product $X = X_1 \times X_2$. By a rectangular set R in X, we mean any set of the form $R = A \times B$ where $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$. We will take as the family of elementary sets for the product measure $\mathcal{E} = \{ E = \bigcup_{j=1}^{n} R_j \; ; \; R_j = A_j \times B_j \; ; \; A_j \in \mathcal{A}_1 , \; B_j \times \mathcal{A}_2 \}$ (*)

where R_j are disjoint rectangles and n is an arbitrary natural number. $\mathcal{E}$ is an algebra.

Def We define the product measure $\mu_1 \otimes \mu_2 \; (E) = \sum_{j=1}^{n} \mu_1(A_j)\mu_2(B_j)$

for each elementary set $E \in \mathcal{E}$ as defined by (*).

⚠ This definition requires justification because the decomposition given in equations is not unique.

$$E = \bigcup_{j=1}^{n} (A_j \times B_j) = \bigcup_{j=1}^{n} (C_j \times D_j)$$

It follows from the finite additivity of each of the measures μ_1 and μ_2 that:

$$\mu_1(A_j)\,\mu_2(B_j) = \sum_{k=1}^{m} \mu_1(A_j \cap C_k)\,\mu_2(B_j \cap D_k)$$

and

$$\mu_1(C_k)\,\mu_2(D_k) = \sum_{j=1}^{n} \mu_1(A_j \cap C_k)\,\mu_2(B_j \cap D_k)$$

$$\Rightarrow \sum_{k=1}^{m} \mu_1(C_k)\,\mu_2(D_k) = \sum_{j=1}^{n} \mu_1(A_j)\,\mu_2(B_j).$$

Lemma 5.2: Suppose $A \times B = \bigcup_{j=1}^{\infty} (A_j \times B_j)$ where $A, A_j \in \mathcal{A}_1$ and $B, B_j \in \mathcal{A}_2$ and the $(A_j \times B_j)$ are disjoint. Then $\mu_1 \otimes \mu_2 (A \times B) = \sum_{j=1}^{\infty} \mu_1 \otimes \mu_2 (A_j \times B_j)$.

Proof: we have: $\chi_A(x)\,\chi_B(y) = \chi_{A \times B}(x,y) = \sum_{j=1}^{\infty} \chi_{A_j}(x)\,\chi_{B_j}(y)$

By the Monotone Convergence Theorem

$$\chi_A(x)\,\mu_2(B) = \sum_{j=1}^{\infty} \chi_{A_j}(x)\,\mu_2(B_j),$$

and also by the Monotone Convergence Thm:

$$\mu_1(x)\,\mu_2(B) = \sum_{j=1}^{\infty} \mu(A_j)\,\mu_2(B_j) \quad \blacksquare$$