Def 3.5: Let f, g be 2 functions defined on $(X, \mathcal{B}, \mu)$.
We say that $f = g$ almost every where, written $f = g$ a.e if $\{x \in X,\ f(x) \neq g(x)\}$ is of measure zero. In particular if A is a measurable subset, then $\chi_A = 0$ a.e $\Leftrightarrow \mu(A) = 0$.

Def 3.6: Let f be a function defined on $(X, \mathcal{B}, \mu)$. We say that f is defined almost everywhere on X if there exist a null subset N such that f is defined on the complementary of N.

Def 3.7: A sequence $(f_n)_n$ of functions defined on $(X, \mathcal{B}, \mu)$ is said that converges almost everywhere to a function f if the set of x where this fails has measure zero.

Prop Let f and g be 2 non-negative measurable functions defined on a measure space $(X, \mathcal{B}, \mu)$.

① $\int_X f(x)\, d\mu(x) = 0$ if and only if $f = 0$ almost everywhere.

② If $f = g$ almost everywhere then $\int_X f(x)\, d\mu(x) = \int_X g(x)\, d\mu(x)$.

Proof:
① we suppose that $\int_X f(x)\, d\mu(x) = 0$. If $A_n = \{x \in X\ ;\ f(x) \geq 1/n\}$ then
$\chi_{A_n} \leq n f$ and $\int_X \chi_{A_n}(x)\, d\mu(x) = \mu(A_n) \leq n \int_X f(x)\, d\mu(x) = 0$

Then $\forall n \in \mathbb{N}$, $\mu(A_n) = 0$. It results that $\{x \in X \,/\, f(x) \neq 0\} = \bigcup_n A_n$ is a null set.
Conversely, if $f = 0$ almost everywhere then for all $n \in \mathbb{N}$, we define $f_n = \inf(f, n)$. The sequence $(f_n)_n$ is increasing and $\int_X f_n(x)\, d\mu(x) = 0$ then it follows by the monotone convergence theorem $\int_X f(x)\, d\mu(x) = 0$.

②* We suppose that $f \leq g$. Then the function $h = g - f$ is defined almost everywhere and equal to 0 almost everywhere.
If $\int_X f(x)\, d\mu(x) = \int_X g(x)\, d\mu(x) = +\infty$, then we have the desired result.
If $\int_X f(x)\, d\mu(x) = \int_X g(x)\, d\mu(x) < \infty$, we have

$$0 = \int_X h(x)\, d\mu(x) = \int_X g(x)\, d\mu(x) - \int_X f(x)\, d\mu(x)$$

- let now define the function $h = \inf(f, g)$. h is non-negative measurable function and we have: $h = f = g$ a.e. Since $h \leq f$ then $\int_X h(x)\, d\mu(x) = \int_X f(x)\, d\mu(x)$ and since $h \leq g$ then $\int_X h(x)\, d\mu(x) = \int_X g(x)\, d\mu(x)$

$$\Rightarrow \quad \int_X f(x)\, d\mu(x) = \int_X g(x)\, d\mu(x)$$

Def 3.9:
Let $f: X \to \overline{\mathbb{R}}$ be a measurable function. If $f^+ = \sup(f, 0)$ and $f^- = \sup(-f, 0)$ then $f = f^+ - f^-$. The function f is called *integrable* by respect to the measure μ if and only if $\int_X f^+(x)d\mu(x)$ and $\int_X f^-(x)d\mu(x)$ are finite.
The integral of f will be denoted $\int_X f(x)d\mu(x) = \int_X f^+(x)d\mu(x) - \int_X f^-(x)d\mu(x)$
We define $L^1(X)$ the space of integrable functions on X.

Prop 3.10: The set $L^1(X)$ is a vector space on $\mathbb{R}$ and the map $f \to \int_X f(x)$ is a linear form on $L^1(X)$ and we have

$$\left|\int_X f(x)d\mu(x)\right| \leq \int_X |f(x)|d\mu(x).$$

proof:
- Let f and g be 2 integrable functions. Since $|f+g| \leq |f| + |g|$ then $\int_X |f+g|\,d\mu \leq \int_X |f|\,d\mu + \int_X |g|\,d\mu$. Then $(f+g)$ is integrable.

We have $f+g = (f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$.

Then $\int_X (f+g)\,d\mu = \int_X (f+g)^+ d\mu - \int_X (f+g)^- d\mu = \int_X f^+ d\mu - \int_X f^- d\mu + \int_X g^+ d\mu - \int_X g^- d\mu$
$= \int_X f\,d\mu + \int_X g\,d\mu$.

- For all $a > 0$ and $b > 0$, we have $|a-b| \leq a+b$.

$$\left|\int_X f(x)\,d\mu(x)\right| = \left|\int_X f^+(x)d\mu(x) - \int_X f^-(x)\,d\mu(x)\right|$$
$$\leq \int_X f^+(x)d\mu(x) + \int_X f^-(x)\,d\mu(x) \quad (|f| = f^+ + f^-)$$
$$\leq \int_X |f(x)|\,d\mu(x)$$

In the case: $f: X \to \mathbb{C} \cup \{\infty\}$, $|f| = \sqrt{(\mathrm{Re} f)^2 + (\mathrm{Im} f)^2}$ is μ-integrable on X.

$$\int_X f(x)\,d\mu(x) = \int_X \mathrm{Re} f(x)\,d\mu(x) + i\int_X \mathrm{Im} f(x)\,d\mu(x)$$

We can prove that $L^1(X)$ is a vector space on $\mathbb{C}$ and the map: $f \to \int_X f(x)d\mu(x)$ is a $\mathbb{C}$-linear form on $L^1(X)$ and we have

$$\left|\int_X f(x)\,d\mu(x)\right| \leq \int_X |f(x)|\,d\mu(x).$$

proof:
$\exists \theta \in \mathbb{R}$, $\int_X f(x)\,d\mu(x) = \left|\int_X f(x)d\mu(x)\right| . e^{i\theta}$.

As $e^{-i\theta}\int_X f(x)\,d\mu(x) \in \mathbb{R}$ then $\left|\int_X f(x)d\mu(x)\right| = \left|e^{-i\theta}\int_X f(x)\,d\mu(x)\right|$
$\overset{\text{by linearity}}{=} \left|\int_X e^{-i\theta} f(x)d\mu(x)\right|$
$= \int_X \mathrm{Re}(e^{-i\theta} f(x))d\mu(x) \leq \int_X |f(x)|d\mu$

Corollary 3.11

① If f is measurable and $a \leq f \leq b$ and $\mu(X) < \infty$, then $f \in L^1(X)$ and we have: $a\,\mu(X) \leq \int_X f(x)\,d\mu(x) \leq b\,\mu(X)$.

* ② if $f: \mathbb{R} \to \overline{\mathbb{R}}$ measurable. Then $\forall a > 0$, $\mu(\{|f| \geq a\}) \leq \frac{1}{a} \int_{\mathbb{R}} |f|\,d\mu$

* ③ If $f: \mathbb{R} \to \overline{\mathbb{R}}$ is μ-integrable then $\mu(\{|f| = +\infty\}) = 0$.

④ If f is measurable and $g \in L^1(X)$ and $f \leq g$ then $\int_X f(x)\,d\mu(x) \leq \int_X g(x)\,d\mu(x)$

⑤ If E is a measurable null set, then $\int_E f(x)\,d\mu(x) = 0$ for any measurable function f.

⑥ Any bounded measurable function and equal to zero in the complementary of a subset of finite measure is integrable.

Proof: ② Tchebychef's inequality: For $a > 0$,

We put $g = a\,\chi_{\{|f| \geq a\}}$; we have $g \leq |f|$ then $\int g\,d\mu \leq \int |f|\,d\mu$

but $\int g\,d\mu = a\,\mu(\{|f| \geq a\})$. So $\mu(\{|f| \geq a\}) \leq \frac{1}{a} \int |f|\,d\mu$.

③ $\forall n \geq 1$, $\mu(\{|f| = +\infty\}) \leq \mu(\{|f| \geq n\}) \leq \frac{1}{n} \int |f|\,d\mu \xrightarrow[n \to \infty]{} 0$

because $f \in L^1(\mathbb{R})$.

⚠ On a measure space $(X, \mathcal{B}, \mu)$, the set of functions that are $f = 0$ a.e is a vector space of $L^1(X, \mathcal{B})$ closed under countable (sup, inf). We denote $L^1(X, \mathcal{B})$ or $L^1(\mu)$ the quotient space $\mathcal{L}^1(X, \mathcal{B})$ by the space of null a.e functions. We call that $f = g$ in $L^1(X)$ if $f = g$ μ-almost everywhere.

Def A sequence $(f_n)_n$ of measurable functions on a measure space (X, μ) converges almost everywhere if the set of divergence of the sequence is a null set. We will denote by $\lim f_n$ any arbitrary measurable function f such that $f_n \xrightarrow[n \to \infty]{} f$ a.e on X.

3.3 Dominate Convergence Theorem

Thm (Dominate Convergence Thm or Lebesgue Thm)

Let $(f_n)_n$ be a sequence of measurable functions on a measure space $(X, \mathcal{B}, \mu)$. We assume that:

ⓘ The sequence $(f_n)_n$ converges a.e on X to a measurable function f definite a.e

ⓘⓘ $\exists$ a non-negative integrable function g such that: $|f_n| \leq g$ a.e $\forall n$

Then the sequence $(f_n)_n$ and the function f are integrable and we have: $\int_X f\,d\mu = \lim_{n \to \infty} \int_X f_n\,d\mu$

The interest of the dominated Convergence Thm is that it does not require uniform convergence to permute the limit and the integral.

Thm Let $(f_n)_n$ be a sequence of measurable functions on a measure space $(X, \mathcal{B}, \mu)$. We assume that there exist a non-negative integrable function g such that for all n, $|f_n| \le g$ a.e. Then

$$\int_X \underline{\lim}\, f_n(x)\, d\mu(x) \le \underline{\lim} \int_X f_n(x)\, d\mu(x)$$

$$\overline{\lim} \int_X f_n(x)\, d\mu(x) \le \int_X \overline{\lim}\, f_n\, d\mu(x)$$

and if the sequence $(f_n)_n$ converges a.e on X to a measurable function f defined a.e, then $f \in L^1(X)$ and we have:

$$\int_X f(x)\, d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)\, d\mu(x)$$

Proof: The function g is finite almost everywhere on X because it is integrable. If we replace g by the function $g\chi_{\{x / g(x) < +\infty\}}$ this which not change the inequalities $|f_n| \le g$ a.e
Thus we can suppose that g is finite on X. We replace the sequence

$(f_n)_n$ by the functions $f_n \chi_{\{|f_n| \le g\}}$, this which not modified the integrals $\int_X f_n(x)\, d\mu(x)$ neither the equivalence classes $\lim\limits_{n\to\infty} f_n$ almost everywhere. Then we can suppose that $|f_n| \le g$ on X.
From these modifications, the functions $(f_n)_n$, $\overline{\lim}\, f_n$ and $\underline{\lim}\, f_n$ are finite and integrable on X. We apply the Fatou's lemma to the sequence $f_n + g$ we shall have:

$$\int_X \underline{\lim}\, (f_n+g)(x)\, d\mu(x) \le \underline{\lim} \int_X (f_n+g)(x)\, d\mu(x)$$

Since $\underline{\lim}\,(f_n+g) = (\underline{\lim}\, f_n) + g$ on X, we shall have:

$$\int_X \underline{\lim}\, f_n(x)\, d\mu(x) \le \underline{\lim} \int_X f_n(x)\, d\mu(x).$$

And from Fatou's lemma applied to the sequence $(-f_n+g)_n$ we shall have:

$$\int_X \underline{\lim}\,(-f_n)(x)\, d\mu(x) \le \underline{\lim} \int_X -f_n(x)\, d\mu(x)$$

Then

$$\overline{\lim} \int_X f_n(x)\, d\mu(x) \le \int_X \overline{\lim}\, f_n(x)\, d\mu(x).$$

examples:

① Let f be an integrable function on $[0,1]$ then $\lim\limits_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0$

In fact: $|x^n f(x)| \leq |f(x)|$ which is integrable.

$$\lim_{n\to\infty} x^n f(x) = 0$$

By Dominate Convergence Thm, we have $\lim\limits_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0.$

② Let (f_n) be the sequence defined in $[0,1]$ by $f_n(x) = \dfrac{nx}{1+n^4x^4}$

$f_n \xrightarrow[n\to\infty]{\text{cvg uniformly}} 0$. By Dominate Convergence Thm,

$$\lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^4x^4}\,dx = 0.$$

③ Let $(f_n)_n$ be the sequence defined in $[0,1]$ by $f_n(x) = nx\,e^{-nx}\chi_{[0,1]}(x)$

$f_n \xrightarrow[n\to\infty]{} 0$ and $0 \leq f_n \leq \chi_{[0,1]}$. By Dominate Convergence Thm,

$$\lim_{n\to\infty} \int_0^1 nx\,e^{-nx}\,dx = 0$$

④ $\left(f_n(x) = nx^2 e^{-nx^2}\chi_{[0,1]}(x)\right)_n$ be a sequence. $f_n \xrightarrow[n\to\infty]{} 0$ and

$0 \leq f_n \leq \chi_{[0,1]}$ Then by Dominate convergence Thm, $\lim\limits_{n} \int_0^1 nx^2 e^{-nx^2}\,dx = 0$

⚠ expl ② $f_n(x) = \dfrac{nx}{1+n^4x^4}$; $x \in [0,1]$

$\sup\limits_{[0,1]} f_n(x) = ?$

$$f_n'(x) = \frac{n(1+n^4x^4) - nx\cdot 4n^4x^3}{(1+n^4x^4)^2} = \frac{n+n^5x^4-4n^5x^4}{(1+n^4x^4)^2} = \frac{n-3n^5x^4}{(1+n^4x^4)^2}$$

$$n-3n^5x^4 = n(1-3n^4x^4) = n(1-\sqrt{3}n^2x^2)(1+\sqrt{3}n^2x^2)$$
$$= n(1-3^{1/4}nx)(1+3^{1/4}nx)(1+\sqrt{3}n^2x^2)$$

$f_n'(x) = 0 \iff 1-3^{1/4}nx = 0 \iff \dfrac{1}{3^{1/4}n} = x$; $\alpha = 3^{1/4}$

$$\begin{array}{c|ccccc|}
x & 0 & & 1/\alpha n & & 1 \\
\hline
f_n'(x) & & + & 0 & - & \\
\hline
f_n(x) & 0 & \nearrow & & \searrow & n/(1+n^4) \\
\end{array}$$

$$f_n\left(\frac{1}{\alpha n}\right) = \frac{n\frac{1}{\alpha n}}{1+n^4\cdot\frac{1}{\alpha^4 n^4}} = \frac{\alpha^3}{\alpha^4+1} = \underline{\underline{cst}} = \frac{3^{3/4}}{4}$$

$f_n(x) \xrightarrow[n\to\infty]{} 0$ a.e

$f_n(x) \leq cst$ (which is integrable on $[0,1]$.

Dominate Convergence thm

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^4x^4}\,dx = \int_0^1 0\,dx = 0.$$

2^{sd} method:

$$\lim_{n\to\infty}\int_0^1 \frac{nx}{1+n^4x^4}\,dx = \lim_{n\to\infty}\frac{1}{n}\int_0^n \frac{t}{1+t^4}\,dt = 0$$

$t = nx$, $dt = n\,dx$, because $\int_0^\infty \frac{t}{1+t^4}\,dt < \infty$.

---

* Now we take: $f_n(x) = \dfrac{nx}{1+n^2x^4}$ on $[0,1]$; $\lim_{n\to\infty} f_n(x) = 0$ a.e

$\sup_{[0,1]} f_n(x) =$

$$f_n'(x) = \frac{n(1+n^2x^4) - nx\cdot 4n^2x^3}{(1+n^2x^4)^2} = \frac{n+n^3x^4-4n^3x^4}{(1+n^2x^4)^2}$$

$$= \frac{n-3n^3x^4}{(1+n^2x^4)^2} = \frac{n(1-3n^2x^4)}{(1+n^2x^4)^2} = \frac{n(1-\sqrt{3}\,nx^2)(1+\sqrt{3}\,nx^2)}{(1+n^2x^4)^2}$$

$$1-\sqrt{3}\,nx^2 = (1-3^{1/4}\sqrt{n}\,x)(1+3^{1/4}\sqrt{n}\,x)$$

$f_n'(x) = 0 \iff x_0 = \dfrac{1}{\alpha\sqrt{n}}$; $\alpha = 3^{1/4}$

x	0		x_0		1
f_n'		$+$		$-$	
$f_n(x)$	0	$\nearrow$		$\searrow$	$\frac{n}{1+n^2}$

; $f_n(x_0) = \dfrac{n\cdot\frac{1}{\alpha\sqrt{n}}}{1+n^2\frac{1}{\alpha^4n^2}} = \dfrac{\sqrt{n}}{\alpha(1+1/\alpha^4)} \xrightarrow[n\to\infty]{} \infty$

So $\nexists\, g$ integrable / $|f_n| \le g$ ae

It follows that $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{nx}{1+n^2x^4}\,dx \neq \int_0 \Big(\underbrace{\lim_{n\to\infty}\frac{nx}{1+n^2x^4}}_{0 \text{ a.e}}\Big)\,dx$

$t = nx^2$

$dt = 2nx\,dx$

$$\int_0^1 \frac{nx}{1+n^2x^4}\,dx = \frac{1}{2}\int_0^n \frac{dt}{1+t^2} = \frac{1}{2}\tan^{-1}(n)$$

$$\lim_{n\to\infty}\int_0^1 \frac{nx}{1+n^2x^4}\,dx = \frac{1}{2}\lim_{n\to\infty}\tan^{-1}(n) = \frac{\pi}{4}.$$

3.4 Applications

If we apply the Dominate Convergence Theorem on the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ with the measure μ defined by: $\mu(n) = 1$ for all $n \in \mathbb{N}$, we have the following result:

Theorem:

Let $(a_{m,n})_{m,n}$ be a double sequence of complex numbers such that:

(i) $\lim_{n\to\infty} a_{m,n} = a_m$ for all $m \in \mathbb{N}$.

(ii) There exit a sequence $(b_m)_m$ of non-negative real numbers such that $\sum_m b_m < \infty$ and $|a_{m,n}| \leq b_m$ for all $n \in \mathbb{N}$.

Then we have: $\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} a_m$.

* Integral Depending on Parameter

Let $(X, \mathcal{B}, \mu)$ be a measure space and let E be a metric space:

Prop Let $f : E \times X \to \mathbb{C}$ such that for all $t \in E$; the mapping $x \to f(t,x)$ is integrable. we define $F(t) = \int_X f(t,x)\, d\mu(x)$

Let $a \in E$, we assume that:

For almost any $x \in X$; the mapping $t \to f(t,x)$ is continuous in a.

There exit a neighborhood $V(a)$ of a and an integrable function g such that $\forall t \in V(a)$; $|f(t,\cdot)| \leq g(\cdot)$. Then F is continuous in a.

Proof: Let (a_n) be a sequence in $V(a)$ which converges to a.

Consider the sequence $(f(a_n,\cdot))_n$

(i) $(f(a_n,\cdot))_n$ converges a.e on X to $f(a,\cdot)$

(ii) $|f(a_n,\cdot)| \leq g(\cdot)$ a.e $\forall n$

Then by Dominate Convergence Thm:

$$\int_X f(a,x)\, d\mu(x) = \lim_{n\to\infty} \int_X f(a_n,x)\, d\mu(x)$$

$$\left(\text{ie: } F(a) = \lim_{n\to\infty} F(a_n)\right) \quad \blacksquare$$

So F is continuous in a

Prop Let Ω be an open set of $\mathbb{R}$ (resp $\mathbb{C}$). Let $f : \Omega \times X \to \mathbb{C}$ such that for all $t \in \Omega$, the mapping $x \to f(t,x)$ is integrable

We define $F(t) = \int_X f(t,x)\, d\mu(x)$. we assume that:

- For almost all $x \in X$; $t \to f(t,x)$ is differentiable on Ω (resp holomorphic on Ω). We denote $\frac{\partial f}{\partial t}(t,x)$ its derivative.
- The function $f(t,\cdot)$ is integrable on X and $\exists$ a non-negative integrable functiong such that for almost all $x \in X$, $\left|\frac{\partial f}{\partial t}(t,x)\right| \leq g(x)$ for any $t \in \Omega$.

Then F is differentiable on Ω (resp holomorphic) and for any t in Ω,

$$\frac{d}{dt}\int_X f(t,x)\,d\mu(x) = \int_X \frac{\partial}{\partial t} f(t,x)\,d\mu(x)$$

Proof: let $a \in \Omega$, and (h_n) be a sequence of real numbers converging to 0 and such that $a+h_n \in \Omega$ ($h_n \neq 0\ \forall n$).

We define the sequence $(\varphi_n)_n$ by $\varphi_n(x) = \dfrac{f(a+h_n, x) - f(a,x)}{h_n}$

- For almost all $x \in X$, $\lim_{n\to\infty} \varphi_n(x) = \dfrac{\partial f}{\partial t}(a,x)$ and for such x we have $|\varphi_n(x)| \leq g(x)$.

By Dominate Convergence Thm the function $\frac{\partial}{\partial t} f(t,x)$ is integrable and

$$\int_X \frac{\partial f}{\partial t}(a,x)\,d\mu(x) = \lim_{n\to\infty}\int_X \varphi_n(x)\,d\mu(x) = \lim_{n\to\infty}\frac{F(a+h_n)-F(a)}{h_n}$$

So $$F'(a) = \int_X \frac{\partial f}{\partial t}(a,x)\,d\mu(x) \quad \blacksquare$$