1. Lebesgue Integral

1- Simple Functions:

Def: Let $(X, \mathcal{A})$ be a measurable space. A function $f: X \to \mathbb{R}$ (or $\overline{\mathbb{R}}$) is called a <u>simple function</u> if it is measurable and takes a finite number of values.

- Let $f: X \longrightarrow \overline{\mathbb{R}}$ be a simple function. If $\{c_1, \dots, c_m\}$ is the set of values of f: $c_j \neq c_k$ for $j \neq k$, and $A_j = \{x \in X$ such that $f(x) = c_j\}$ then $X = \bigcup_j A_j$. $A_j \cap A_k = \emptyset$ if $j \neq k$ and $f = \sum_{j=1}^{m} c_j \chi_{A_j}$.

⚠ f measurable if and only if A_j is measurable $\forall j$.

Thm: Let $(X, \mathcal{A})$ be a measurable space and $f: X \longrightarrow \overline{\mathbb{R}}$:

① If f is a measurable and bounded, there exists a sequence $(f_n)_n$ of simple functions which converges uniformly on X to f.

② If f is a non-negative measurable function. Then there exists a sequence of non negative measurable simple functions which increases to f.

Proof: ① Let $M > 0$ such that $\forall x \in X$, $|f(x)| < M$. For $(n, k) \in \mathbb{N} \times \mathbb{Z}$ and $-2^n \leq k \leq 2^n - 1$; we set $A_{n,k} = \{x \in X : \frac{kM}{2^n} \leq f(x) < \frac{(k+1)M}{2^n}\}$.

and we define f_n by: $f_n = \sum_{k=-2^n}^{2^n-1} \frac{kM}{2^n} \chi_{A_{n,k}}$. The sets $A_{n,k}$ are measurables and f_n is measurable, $\forall n$.

For $x_0 \in X$, $\exists\, k_0$ s.t $x_0 \in A_{n,k_0}$. Then $f_n(x_0) = \frac{Mk_0}{2^n}$ and $|f(x_0) - f_n(x_0)| < \frac{M}{2^n}$. Then $f_n \xrightarrow[n\to\infty]{} f$ uniformly on X.

② For $n \in \mathbb{N}$, let $g_n = \inf(f, n) - 1/n$. g_n is bounded measurable then from ①, $\exists (f_m)$ a sequence of simple measurable functions such that $\| f_m - g_n \|_\infty = \sup_{m,x} |f_m(x) - g_n(x)| < 1/2^n$.

$$\lim_{n\to\infty} f_m = \lim_{n\to\infty} g_n = \lim_{n\to\infty} \inf(f, n) = f.$$

$$f_n \leq g_n + \frac{1}{2^n} = \inf(f, n) - 1/n + 1/2^n \leq \inf(f, n+1) - \frac{1}{n+1} + \frac{1}{2^{n+1}}$$
$$\leq f_{n+1}. \quad \text{(provided } n \text{ big)}$$

So $(f_n) \nearrow$.

2- Integration:

For constructing the integral of real measurable functions on a measure space $(X, \mathcal{B}, \mu)$, we proceed by steps: We begin by the case of the integral of simple functions, then we define the integral of non negative measurable functions by the increasing limit and we show that the monotone limit allows to define the integral of the measurable non-negative functions, and finally the decomposition of a measurable arbitrary functions: $f = \max(f, 0) - \max(-f, 0) = f^+ - f^-$.

as the difference of 2 measurable non-negative functions extends the definition of the integral to the measurable functions.

Def: If $f = \sum_{k=1}^{N} \lambda_k \chi_{\{x\in X / f(x) = \lambda_k\}}$ is a non-negative measurable simple function, we define the integral of f by

$$\int_X f(x)\, d\mu(x) = \sum_{k=1}^{N} \lambda_k\, \mu(\{x\in X / f(x) = \lambda_k\})$$

In particular, if $f = \chi_A$, A is measurable subset then $\int_X f(x)\, d\mu(x) = \int_X \chi_A(x)\, d\mu(x) = \mu(A)$. with the convention that $0\times(+\infty) = 0$.

Prop: Let $\mathcal{E}^+$ be the cone of non-negative simple functions on the measure space $(X, \mathcal{B}, \mu)$. The integral defined on $\mathcal{E}^+$ have the following properties:

1. $\forall \alpha \in \mathbb{R}_+$, $\forall f \in \mathcal{E}^+$; $\int_X \alpha f(x)\, d\mu(x) = \alpha \int_X f(x)\, d\mu(x)$.
2. $\forall f, g \in \mathcal{E}^+$; $\int_X (f+g)(x)\, d\mu(x) = \int_X f(x)\, d\mu(x) + \int_X g(x)\, d\mu(x)$
3. $\forall f, g \in \mathcal{E}^+$ such that $f \leq g$; $\int_X f(x)\, d\mu(x) \leq \int_X g(x)\, d\mu(x)$
4. If $(f_n)_n$ is an increasing sequence in $\mathcal{E}^+$ and if f is the limit of the sequence $(f_n)_n$ belongs to $\mathcal{E}^+$ then

$$\int_X f(x)\, d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)\, d\mu(x).$$

Proof: ② $f = \sum_{a\in F} a\, \chi_{\{f=a\}}$; $g = \sum_{b\in G} b\, \chi_{\{g=b\}}$; F (resp G) is the set of values of f (resp of g)

We have For $a \in F$, $\{f=a\} = \bigcup_{b\in G} \{f=a, g=b\}$.

For $b\in G$, $\{g=b\} = \bigcup_{a\in F} \{f=a, g=b\}$.

$$\int_X f(x)\,d\mu(x) = \sum_{a\in F} a\,\mu(\{f=a\}) = \sum_{F\times G} a\,\mu(\{f=a, g=b\})$$

$$\int_X g(x)\,d\mu(x) = \sum_{b\in G} b\,\mu(\{g=b\}) = \sum_{F\times G} b\,\mu(\{f=a, g=b\})$$

$$\Rightarrow \int_X f(x)\,d\mu(x) + \int_X g(x)\,d\mu(x) = \sum_{(a,b)\in F\times G} (a+b)\,\mu(\{f=a, g=b\})$$

As $\{f+g=u\} = \bigcup_{\substack{(a,b)\in F\times G\\ a+b=u}} \{f=a; g=b\}$ then $\mu(\{f+g=u\}) = \sum_{a+b=u} \mu(\{f=a, g=b\})$

So $$\int_X f(x)\,d\mu(x) + \int_X g(x)\,d\mu(x) = \sum_u u\,\mu(\{f+g=u\})$$
$$= \int_X (f+g)(x)\,d\mu(x).$$

③ If $\int_X f(x)\,d\mu(x) \overset{+}{=}\infty$ then $\int_\infty g(x)\,d\mu(x) \overset{X}{=}\infty$. The result is evident if $\int_X f(x)\,d\mu(x) < \infty$ and $\int_X g(x)\,d\mu(x) = +\infty$.

Assume now that: $\int_X f(x)\,d\mu(x) < \infty$ and $\int_X g(x)\,d\mu(x) < \infty$:

Then $\mu(\{x\in X / f(x)=\infty\}) = 0$ and $\mu(\{x\in X, g(x)=\infty\}) = 0$

– Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ the sets of finite values of f resp g.

Set $\tilde{f} = \sum_{j=1}^n a_j\,\chi_{\{f=a_j\}}$ and $\tilde{g} = \sum_{j=1}^n b_j\,\chi_{\{g=b_j\}}$

then $\int_X \tilde{f}(x)\,d\mu(x) = \int_X f(x)\,d\mu(x)$ and $\int_X \tilde{g}(x)\,d\mu(x) = \int_X g(x)\,d\mu(x)$

As $f\le g$ then $h = \tilde{g}-\tilde{f} \in \mathcal{E}^+$.

by ② $$\int_X g(x)\,d\mu(x) = \int_X f(x)\,d\mu(x) + \int_X h(x)\,d\mu(x) \geqslant \int_X f(x)\,d\mu(x).$$

④ We need the following lemma:

– <u>Lemma</u>: let $(f_n)_n$ be an increasing sequence in $\mathcal{E}^+$, and if $g\in\mathcal{E}^+$ such that $g \le \lim_{n\to\infty} f_n$ then
$$\int_X g(x)\,d\mu(x) \le \lim_{n\to\infty} \int_X f_n(x)\,d\mu(x)$$

<u>Proof of lemma</u>:

For $y \in g(X)$; Let $E_y = \{x \in X ; g(x) = y\}$. To prove the lemma it suffices to prove that: $\forall y \in g(X)$

$$\int_X g(x)\, \chi_{E_y}(x)\, d\mu(x) = y\, \mu(E_y) \leq \lim_{n\to\infty} \int_X f_n(x) \chi_{E_y}(x) d\mu$$

- If $y=0$, the result is evident.
- For $0<t<y$, we put: $A_n = E_y \cap \{x \in X \,/\, f_n(x) \geq t\}$

(A_n) is an increasing sequence of measurable sets and $E_y = \lim_{n\to\infty} A_n$ because for all $x \in E_y$: $f_n(x) > t$ for n large.

$$t\,\mu(\{E_y \cap \{x \in X : f_n(x) > t\}\}) = \int_X t\, \chi_{E_y \cap \{x / f_n(x) > t\}}(x)\, d\mu(x)$$

$$\leq \int_X f_n(x)\, \chi_{E_y}(x)\, d\mu(x)$$

$$\Rightarrow t\,\mu(E_y) \leq \lim_{n\to\infty} \int_X f_n(x) \chi_{E_y}(x)\, d\mu(x).$$ This is for any $0<t<y$

so $$y\,\mu(E_y) \leq \lim_{n\to\infty} \int_X f_n(x)\, \chi_{E_y}(x)\, d\mu(x).$$

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to prove ④: we denote $g = \lim_n f_n$. Then $f_n \leq g$ $\forall n \in \mathbb{N}$, and the increasing sequence $\left(\int_X f_n(x) d\mu(x)\right)_n$ is bounded above by $\int_X g(x)\, d\mu(x)$. So

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) \leq \int_X g(x)\, d\mu(x).$$

Now by lemma, we have:
$(g \leq \lim_n f_n)$

$$\int_X g(x)\, d\mu(x) \leq \lim_{n\to\infty} \int_X f_n(x)\, d\mu(x).$$

We conclude that

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) = \int_X \left(\lim_{n\to\infty} f_n(x)\right) d\mu(x).$$

Def: Let f be a non-negative measurable function on a measure space $(X, \mathcal{B}, \mu)$, we define: $\int_X f(x) d\mu(x) = \sup\left\{\int_X g(x) d\mu(x) : \begin{array}{l} g \leq f \\ g \in \mathcal{E}^+ \end{array}\right\}$
this is a non-negative number finite or infinite.

⚠ If f is non-negative measurable function on $(X, \mathcal{B}, \mu)$, the Thm 1.2 yields the existence of an increasing sequence (f_n) of $\mathcal{E}^+$ which converges to f. Then $$\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) \leq \int_X f(x)\, d\mu(x). \quad (*)$$

In the other hand, for every $g \in \mathcal{E}^+ / \ g \leq f = \lim_n f_n$

We have prooved that $\int_X g(x)\,d\mu(x) \leq \lim_{n\to\infty} \int_X f_n(x)\,d\mu(x)$ (lemma).

So From definition: we get

$$\int_X f(x)\,d\mu(x) \leq \lim_{n\to\infty} \int_X f_n(x)\,d\mu(x). \quad (**)$$

From $(*)$ and $(**)$, we get:

$$\lim_{n\to\infty} \int_X f_n(x)\,d\mu(x) = \int_X \lim_n f_n(x)\,d\mu(x)$$

Thm 2: Let f and g be 2 non negative measurable functions on a measure space $(X, \mathcal{B}, \mu)$ and let $\lambda \geqslant 0$, then we have:

① $\int_X \lambda f(x)\,d\mu(x) = \lambda \int_X f(x)\,d\mu(x)$.

② $\int_X (f+g)(x)\,d\mu(x) = \int_X f(x)\,d\mu(x) + \int_X g(x)\,d\mu(x)$.

③ If $f \leq g$ then $\int_X f(x)\,d\mu(x) \leq \int_X g(x)\,d\mu(x)$.

Proof: Apply prop to (φ_n) and (ψ_n) where (φ_n) is $\nearrow$ converge to f

$\varphi_n, \psi_n \in \mathcal{E}^+$, (ψ_n) is $\nearrow$ —— to g ∎

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3. Convergence Theorems

Thm (Monotone Convergence Thm or Beppo-Lévi Thm)

Let $(f_n)_n$ be an increasing sequence of measurable non-negative functions on a measure space $(X, \mathcal{B}, \mu)$ then

$$\int_X \lim_n f_n(x)\,d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)\,d\mu(x)$$

Proof: For $n \in \mathbb{N}$, $\exists\, (\varphi_{n,j})_j \nearrow$ non negative of $\mathcal{E}^+$ wich converges to f_n.

For any j, we set $\psi_j = \sup_{1 \leq n \leq j} \varphi_{n,j}$.

We have $\psi_j \leq \psi_{j+1}\ \forall j$; $(\psi_j)_j$ is $\nearrow$ of $\mathcal{E}^+$.

We want to prove that $\psi_j \xrightarrow[j\to\infty]{} f$.

For all $j \geqslant n$, $\varphi_{n,j} \leq \psi_j$ then $f_n = \lim_{j\to\infty} \varphi_{n,j} \leq \lim_{j\to\infty} \psi_j$

In other hand, ~~and~~ $f = \lim_{n\to\infty} f_n \leq \lim_{j\to\infty} \psi_j$.

$\varphi_{n,j} \leq f_n \leq f$ show that $\psi_j \leq f$ and $\lim_{j\to\infty} \psi_j \leq f$. We deduce $\lim_{j\to\infty} \psi_j = f$. Then $\int_X f(x)\,d\mu(x) = \lim_{j\to\infty} \int_X \psi_j(x)\,d\mu(x)$.

Moreover $\psi_j \leq f_j \Rightarrow \lim_{j\to\infty} \int_X \psi_j(x)\,d\mu(x) \leq \lim_{j\to\infty} \int_X f_j(x)\,d\mu(x) \leq \int_X f(x)\,d\mu(x)$ ∎

Lemma (Fatou's lemma)
Let $(f_n)_n$ be a sequence of non-negative measurable functions on a measure space $(X, \mathcal{B}, \mu)$ then

$$\int \underline{\lim_n}\, f_n \, d\mu(x) \leq \underline{\lim_n} \int f_n \, d\mu$$

$$\underline{\lim}\, f_n = \sup_n \inf_{k \geq n} f_k = \lim_{n \to \infty} \left(\inf_{k \geq n} f_k \right)$$

We have: $\displaystyle\int_X \inf_{k \geq n} f_k(x) \, d\mu(x) \leq \inf_{k \geq n} \int_X f_k(x) \, d\mu(x)$

The result follows from the monotone convergence Thm.

$\varphi_n := \inf_{k \geq n} f_k \qquad \varphi_n \nearrow$ non-negative.

$$\lim_{n \to \infty} \int_X \varphi_n(x) \, d\mu(x) = \int_X \lim_{n \to \infty} \varphi_n(x) \, d\mu(x)$$

$$= \int_X \underline{\lim_n}\, f_n \, d\mu(x)$$

So $\displaystyle\int_X \underline{\lim_n}\, f_n \, d\mu \leq \lim_{n \to \infty} \inf_{k \geq n} \left(\int_X f_k(x) \, d\mu(x) \right) = \underline{\lim_n} \int_X f_n \, d\mu$

Corollary 1 Let $(f_n)_n$ be a sequence of measurable non-negative functions on a measure space $(X, \mathcal{B}, \mu)$ then

$$\int_X \sum_{n=1}^{\infty} f_n(x) \, d\mu(x) = \sum_{n=1}^{\infty} \left(\int_X f_n(x) \, d\mu(x) \right)$$

Corollary 2:

Let $(X, \mathcal{B}, \mu)$ be a measure space and let f be a measurable non-negative function. For all $A \in \mathcal{B}$, let $\tau(A) = \int_X f(x) \chi_A(x) \, d\mu(x)$

Then τ is a non-negative measure on $(X, \mathcal{B})$ called measure of density f by respect to the measure μ.

The integral of a measurable non-negative function g by this measure is given by:

$$\int_X g(x) \, d\tau(x) = \int_X f(x) \, g(x) \, d\mu(x)$$
