6. Lebesgue Measure on $\mathbb{R}$

Thm There exists only and only one measure λ on $\mathcal{B}_{\mathbb{R}}$ satisfying:

i) λ is invariant under translation

$(\forall x \in \mathbb{R}, \forall A \in \mathcal{B}_{\mathbb{R}} \; ; \lambda(x+A) = \lambda(A))$

ii) $\lambda([0,1]) = 1$.

Proof: • **Uniqueness** Assume there exits 2 measures μ and ν on $\mathcal{B}_{\mathbb{R}}$ satisfying (i) and (ii)

$\forall n \geq 1, \nu([0,\frac{1}{n}]) \leq \frac{1}{n} \Rightarrow \nu(\{0\}) = 0$

Then any finite set or countable set is a null set and all intervals $[a,b], (a,b], [b,a)$ have the same measure.

We denote $\mathcal{E}$ the set of finite Union of intervals of $\mathbb{R}$ of the form $[a,b)$, $a<b$. The set $\mathcal{E}$ is closed under finite intersection and $\mathbb{R} = \bigcup_n [-n,n)$. Then $\mu = \nu$ on $\mathcal{E}$.

It follows from thm 5.8 that $\mu = \nu$ on $\mathcal{B}_{\mathbb{R}}$.

• **Existence**: For $A \subset \mathbb{R}$, $\mu^*(A) = \inf_{\mathcal{R}} \sum_{I \in \mathcal{R}} \mathcal{L}(I)$

$\mathcal{R}$: describes the whole of finite or countable coverings of A by open intervals and $\mathcal{L}(I)$ is the length of I.

We prove: • for any interval $I \subset \mathbb{R}$, $\mu^*(I) = \mathcal{L}(I)$.

• for Ω be an open set of $\mathbb{R}$, $\mu^*(\Omega) = \sum_{n=1}^{\infty} \mathcal{L}(I_n)$

• for any sebset $A \subset \mathbb{R}$, $\mu^*(A) = \inf_{O \text{ open} \supset A} \mu^*(O)$

Now we use Caratheodory's thm

The set of the μ^*- mearurable subsets is a σ-algebra $\mathcal{L}$ on $\mathbb{R}$ and $\mu^*|_{\mathcal{L}}$ is a complete measure.

This σ-algebra is called The Lebesgue σ-algebra and the elements of $\mathcal{L}$ are called the lebesgue mearurable sets. we will note $\mathcal{B}^*_{\mathbb{R}}$ this σ-algebra

Proposition: Any Borelian subset is Lebesgue measurable

proof

it suffices to show that $\forall a \in \mathbb{R}$, $(a,\infty) \in \mathcal{L}$,
Let E be a subset of $\mathbb{R}$. Our goal is to prove that

$\mathcal{B}(\mathbb{R})$ $\quad \mu^*(E) = \mu^*(E \cap (a,\infty)) + \mu^*(E \cap (-\infty, a])$

As μ^* is outer measure then $\mu^*(E) \leq \mu^*(E \cap (a,\infty)) + \mu^*(E \cap (-\infty,a])$

If $\mu^*(E) = \infty$ no proof.

Assume $\mu^*(E) < \infty$. Let $\varepsilon > 0$ $\exists\, \Omega_\varepsilon \supset E$ / $\mu^*(\Omega_\varepsilon) \leq \mu^*(E) + \varepsilon$

if $a \notin \Omega_\varepsilon$ $\quad \mu^*(\Omega_\varepsilon) = \sum_{I \in \mathcal{C}} \mathcal{L}(I) = \sum_{I \in \mathcal{C} \cap (a,\infty)} \mathcal{L}(I) + \sum_{I \in \mathcal{C} \cap (-\infty, a)} \mathcal{L}(I)$

with $\mathcal{C}$ the set of connected components of Ω_ε.

$\Rightarrow \mu^*(\Omega_\varepsilon) \geq \mu^*(E \cap (a,\infty)) + \mu^*(E \cap (-\infty, a])$

Now if $a \in \Omega_\varepsilon$. put $\Omega'_\varepsilon = \Omega_\varepsilon \setminus \{a\}$. We remark that $\mu^*(\Omega'_\varepsilon) = \mu^*(\Omega_\varepsilon)$.

Now take $\lambda = \mu^*$. The measure λ on $\mathcal{B}^*_{\mathbb{R}}$ is called the Lebesgue measure on $\mathbb{R}$

Prop: Let $\mathcal{B}^*_{\mathbb{R}}$ the Lebesgue σ-algebra on $\mathbb{R}$,
then $\forall A \in \mathcal{B}^*_{\mathbb{R}}$, $\quad \lambda(A) = \inf_{\omega \text{ open} \supset A} \lambda(\omega)$; $\lambda(A) = \sup_{K \text{ compact} \subset A} \lambda(K)$

We say that the measure λ is *regular*.

Proof: - If A is bounded, $\exists\, n \in \mathbb{N}$ / $A \subset [-n,n]$.
Let $\varepsilon > 0$, $[-n,n] \setminus A$ is measurable, then $\exists$ open set $\omega \supset [-n,n] \setminus A$

such that $\quad \lambda(\omega) \leq \lambda([-n,n] \setminus A) + \varepsilon \qquad \Rightarrow \lambda([-n,n] \setminus A) = \inf_{\omega \supset} \lambda(\omega)$

$\qquad\qquad = \lambda[-n,n] - \lambda(A) + \varepsilon$

Let $K = [-n,n] \cap \omega^c$. K is compact in A.

$$2n = \lambda([-n,n]) = \lambda([-n,n] \cap \omega^c) + \lambda([-n,n] \cap \omega)$$
$$\leq \lambda(K) + \varepsilon + \lambda([-n,n]) - \lambda(A)$$

$\Rightarrow \quad \lambda(A) \leq \lambda(K) + \varepsilon$ and $\lambda(A) = \sup_{K \subset A} \lambda(K)$

- If A is not bounded then $\forall n \in \mathbb{N}$ $\exists\, K_n \underset{\text{compact}}{\subset} [-n,n] \cap A$

$\lambda(K_n) \geq \lambda([-n,n] \cap A) - 1/n$

$\Rightarrow \sup_{K \text{ compact} \subset A} \lambda(K) \geq \sup_n (\lambda(K_n)) \geq \lim_{n \to \infty} (\lambda([-n,n] \cap A) - 1/n) = \lambda(A)$.

7- Measurable functions

Let X and Y be 2 non empty sets. we showed that the pull back of a σ-algebra by a mapping $f: X \to Y$ is a σ-algebra on X.

<u>Def</u>: If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are 2 measurable spaces. A mapping $f: X \to Y$ is called measurable if the σ-algebra $f^{-1}(\mathcal{B}) \subset \mathcal{A}$.

<u>Thm</u> Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be 2 measurable spaces and suppose $\mathcal{B}$ generates the σ-algebra $\mathcal{B}$. A function $f: X \to Y$ is measurable if and only if $\forall V$ in the generator set $\mathcal{B}$, its preimage $f^{-1}(V)$ is in $\mathcal{A}$. $\mathcal{B}$ σ-algebra generated $\mathcal{C}$,

⚠ To show that a mapping $f: X \to Y$ is measurable, it suffices to give a set $\mathcal{C}$ which generates $\mathcal{B}$ and such that $f^{-1}(\mathcal{C}) \subset \mathcal{A}$.

Proposition: Let $(X, \mathcal{A})$ be a measurable space and let $f: X \to \mathbb{R}$ (or in $\overline{\mathbb{R}}$) a mapping. Then f is measurable if one of the following conditions is fullfilled

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① $\forall a \in \mathbb{R}, \{x \in X, f(x) \geq a\} \in \mathcal{A}. \Leftrightarrow f^{-1}(\underbrace{[a, \infty)}_{\mathcal{B}_{\mathbb{R}}}) \in \mathcal{A}$

② $\forall a \in \mathbb{R}; \{x \in X ; f(x) < a\} \in \mathcal{A}.$

③ $\forall a \in \mathbb{R}: \{x \in X : f(x) \leq a\} \in \mathcal{A}$

④ $\forall a, b \in \mathbb{R}; \{x \in X; a < f(x) < b\} \in \mathcal{A}. \Leftrightarrow f^{-1}(a, b) \in \mathcal{A}.$

⑤ $\forall a, b \in \mathbb{R}; \{x \in X; a \leq f(x) < b\} \in \mathcal{A}.$

The space $\mathbb{R}$ (resp $\overline{\mathbb{R}}$) is equipped with the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ (resp $\mathcal{B}_{\overline{\mathbb{R}}}$). We take the measurable spaces $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and $(\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$.

⚠ Let X and Y 2 topological spaces and let $\mathcal{B}_X$ and $\mathcal{B}_Y$ the Borelian σ-algebras on X and Y respectively. Then every <u>continuous function</u> is <u>measurable</u>.

Every measurable function $f: X \to Y$ is called a Borealian function.

Then p_j is measurable.
We have $f_j = p_j \circ f$ is measurable if f measurable (by prop 7.4).
Now we suppose that f_j, $j=1,\dots,n$ are measurable.
Let $A_1 \times \dots \times A_n$ be a rectangle in $\prod_{k=1}^{n} X_k$ then

$$f^{-1}(A_1 \times \dots \times A_n) = f^{-1}\Big(\bigcap_{j=1}^{n} p_j^{-1}(A_j)\Big) = \bigcap_{j=1}^{n} f^{-1}(p_j^{-1}(A_j))$$
$$= \bigcap_{j=1}^{n} f_j^{-1}(A_j).$$

So f is measurable.

⚠ . $(X, \mathcal{B})$ measurable space, f and g are 2 measurable functions on X with values in $\mathbb{R}$ or $\overline{\mathbb{R}}$. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be continuous. $h := F(f, g)$
Then the function h is measurable.

- let $(X, \mathcal{B})$, $(Y, \mathcal{B}')$ and $(Z, \mathcal{C})$ 3 measurable spaces and let $f : X \times Y \to Z$ a mapping.
Then for $\forall a \in X$ (resp $b \in Y$), the partial mapping $f_a = f(a, \cdot)$ (resp $f_b = f(\cdot, b)$) is measurable.
- Let $(X_1, \mathcal{B}_1), \dots, (X_n, \mathcal{B}_n)$; n measurable spaces. $f_j : X_j \to \overline{\mathbb{R}}$ and $f : \prod_{j=1}^{n} X_j \to \overline{\mathbb{R}}$ defined by $f(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n)$
Assume $f_i \neq 0$, f is measurable $\Leftrightarrow$ $f_1, \dots, f_n$ are measurable

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Proposition: Let $(X_0, \mathcal{B}_0)$, $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ three measurable spaces. Let $f_1 : X_0 \to X_1$ and $f_2 : X_1 \to X_2$ 2 measurable mappings, then the mapping $f_2 \circ f_1$ is measurable.

Proof.
$$(f_2 \circ f_1)^{-1}(\mathcal{B}_2) = f_1^{-1}(f_2^{-1}(\mathcal{B}_2)) \subset f_1^{-1}(\mathcal{B}_1) \subset \mathcal{B}_0 \blacksquare$$

Proposition

Let $(X, \mathcal{B})$ and $(X_j, \mathcal{B}_j)$, $j = 1, \dots, n$. $(n+1)$ measurable spaces. and let $f : X \longrightarrow X_1 \times X_2 \times \dots \times X_n = \prod_{j=1}^{n} X_j$, a mapping $f = (f_1, \dots, f_n)$. Then f is measurable $\Leftrightarrow$ $\forall j$, f_j is measurable $(f_j : X \longrightarrow X_j)$.

Proof. Let $p_j : \prod_{k=1}^{n} X_k \longrightarrow X_j$ natural projection
$x = (x_1, \dots, x_n) \longrightarrow x_j$

$$p_j^{-1}(A_j) = X_1 \times \dots \times A_j \times X_{j+1} \times \dots \times X_n.$$
which is measurable if A_j is measurable.

Prop 7.9: Let $(X, \mathcal{B})$ be a measurable space.

a) If f is measurable of $(X, \mathcal{B})$ with values in $\mathbb{R}$ or $\overline{\mathbb{R}}$, then $|f|$ is measurable.

b) If (f_n) is a sequence of measurable functions on $(X, \mathcal{B})$ with values in $\mathbb{R}$ or in $\overline{\mathbb{R}}$, then the functions g, h, k defined by:

$$g(x) = \sup_n f_n(x) \; ; \; h(x) = \overline{\lim_n} f_n(x) \; ; \; k(x) = \underline{\lim} f_n(x)$$

are measurable.

Proof: a) If $a<0$: $\{x \in X : |f(x)| > a\} = X \in \mathcal{B}$.

If $a \geq 0$; $\{x \in X ; |f(x)| > a\} = \{x \in X, f(x) > a\} \cup \{x \in X, f(x) < -a\}$
$= f^{-1}((a, \infty)) \cup f^{-1}((-\infty, -a)) \in \mathcal{B}$.

So $|f|$ is measurable.

b)
- $\{x \in X, g(x) > a\} = \bigcup_{n \in \mathbb{N}} \{x \in X ; f_n(x) > a\} \in \mathcal{B}$. So g is measurable
- $h(x) := \overline{\lim_n} f_n(x) = \inf_n \sup_{k \geq n} f_k(x)$

$$\{x \in X, h(x) > a\} = \bigcap_{n \in \mathbb{N}} \left(\bigcup_{k \geq n} \{x \in X, f_k(x) > a\} \right) \in \mathcal{B}$$

- $k(x) : \underline{\lim_n} f_n(x) = \sup_n \inf_{k \geq n} f_k(x)$

$$\{x \in X, k(x) > a\} = \bigcup_{n \in \mathbb{N}} \left(\bigcap_{k \geq n} \{x \in X ; f_k(x) > a\} \right) \in \mathcal{B}.$$

A . If f measurable then $f^+ = \sup(f, 0)$ and $f^- = \inf(f, 0)$ are measurable.

- If (f_n) is a sequence of measurable functions which converges pointwise toward a function f on X, then f is measurable.

- For any sequence $(f_n)_n$ of measurable functions with real values on a measurable space X, if $C = \{x \in X, \lim_{n \to \infty} f_n(x) \text{ exists in } \overline{\mathbb{R}}\}$ is measurable.

Proof.

We put $D = C^c$

$D = \{x \in X : \underline{\lim} f_n < \overline{\lim} f_n\}$

we put $g = \underline{\lim} f_n$ and $h = \overline{\lim} f_n$

$D = \bigcup_{r \in \mathbb{Q}} D_r$ where $D_r = \{x \in X / g(x) < r < h(x)\}$

So D is measurable set. then C is measurable set.

Thm: 7.11: Let $A \subset \mathbb{R}^{n}$ and $f : A \longrightarrow \mathbb{R}^{n}$ a mapping. Assume that for any point $a \in A$, there exists a neighborhood $V(a)$ such that $\mu^*(f(A \cap V(a))) = 0$ then $\mu^*(f(A)) = 0$

Proof: For $a \in A$, $\exists$ a Ball $B \subset \mathbb{R}^m$ of center of rational coordinates such that $a \in B$ and $\mu_n^*(f \cap B) = 0$. The family $\mathcal{B}$ of these balls is a least countable and cover A.

It follows that $f(A)$ is covered by the sequence $f(A \cap B)$, $B \in \mathcal{B}$ and every one is of measure zero. It follows $\mu_n^*(f(A)) = 0$.

Thm 7.12

Let $A \subset \mathbb{R}^m$ and $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ a mapping such that:

$\exists$ $s \geqslant \frac{m}{n}$ and $|f(x) - f(y)| \leq M^s |x-y|^s$ $\forall x, y \in A$. $(*)$

Then

① If $s > \frac{m}{n}$ then $\mu_n^*(f(A)) = 0$.

② If $s = \frac{m}{n}$ then $\mu_n^*(f(A)) \leq 2^n (M\sqrt{m})^m \mu_m^*(A)$.

Proof: • Let (P_k) be a covering of A by rectangles of length of it sides $\leq \varepsilon$. $(0 < \varepsilon < 1)$

We assume that for $k \in \mathbb{N}$, $P_k \cap A \neq \emptyset$. Let $a, b \in A \cap P_k$ and $r < \varepsilon < 1$

• We have $\|x-b\|_\infty \leq r/2$, $\|a-b\|_\infty \leq r/2$, $\|x-a\|_\infty \leq r$, P_k is a rectangle centered at a and of radius r.

From $(*)$ $\|f(x) - f(a)\|_\infty \leq (M\sqrt{m})^s r^s$ and $\mu_n^*(f(A \cap P_k)) \leq 2^n (M\sqrt{m})^{ns} r^{ns-m} \cdot r^m$

then $\boxed{\mu_n^*(f(A)) \leq 2^n (M\sqrt{n})^{ns} \varepsilon^{ns-m} \underbrace{\sum_k vol(P_k)}_{\mu_m^*(A)}}$ Vol: volume

① if $s > m/n$ the $ns - m > 0$; when $\varepsilon \to 0$; $\mu^*(f(A)) = 0$.

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② If $s = \frac{m}{n}$; then $ns - m = 0$ so $\mu_n^*(f(A)) \leq 2^n (M\sqrt{n})^m \mu_m^*(A)$.

(Here: $\|x\|_\infty = \sup_j |x_j|$

We have: $\|x\|_\infty \leq |x| \leq \sqrt{n} \, \|x\|_\infty$ $\forall x \in \mathbb{R}^n$)

and $\boxed{A \cap P_k \subset P_k(a, r) \text{ then } f(A \cap P_k) \subset P_k(f(a), 2(M\sqrt{m})^s r^s)}$

⚠ ① Every null set in $\mathbb{R}^n$ is of measure zero in any system of coordinate in $\mathbb{R}^n$, because

If f is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, we have $\|f(x)\| \leq M \|x\|$.

If N is a null set, $\mu_n^*(f(N)) \leq 2^n (M\sqrt{n})^n \underbrace{\mu_n^*(N)}_{0}$.

② Every subspace of dimension $m < n$ is a null set in $\mathbb{R}^n$ because: $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ linear.

As $n > m$, $\mu_n^*(f(\mathbb{R}^m)) = 0$ by ① of thm.

<u>More generally</u>: $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is of class C^1 in any point $a \in A \subset \mathbb{R}^m$. Then $\|f(x) - f(y)\| \leq (1 + \|df(a)\|) \|x-y\|$

If $n > m$, we deduce $\mu_n^*(f(A)) = 0$.

Thm (Egoroff)

Let $(X, \mathcal{B}, \mu)$ be a measure space. Assume that the measure μ is bounded. Let $(f_n)_n$ be a sequence of real or complex measurable functions on X which converges pointwise on X to a function f. For any $\varepsilon > 0$, $\exists A_\varepsilon \in \mathcal{B}$ s.t $\mu(A_\varepsilon) < \varepsilon$ and the restriction of the sequence (f_n) on A_ε^c is uniformly convergent.

Proof $\forall x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$, f is measurable function.

For k, n, $E_n^{(k)} = \bigcap_{p \geq n}^{\infty} \{x / \; |f_p - f| \leq 1/k\}$ is measurable

$\lim_{n\to\infty} E_n^{(k)} = X$

μ bounded $\Rightarrow \lim_{n\to\infty} \mu((E_n^{(k)})^c) = 0 \Rightarrow \exists n_{k,\varepsilon} /$

$\mu((E_{n_{k,\varepsilon}}^{(k)})^c) \leq \varepsilon/2^k \quad \forall k$

$A_\varepsilon = \bigcup_{k=1}^{\infty} (E_{n_{k,\varepsilon}}^{k})^c$. $\quad \mu(A_\varepsilon) \leq \varepsilon \quad \forall \varepsilon$ ∎

Thm (Continuity of measure):

Let μ be a measure on $(X, \mathcal{A})$. If $(A_j)_j$ is an increasing sequence in $\mathcal{A}$ and $A = \bigcup_{j=1}^{\infty} A_j$ then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$

Proof 1) If for some n_0, $\mu(A_{n_0}) = +\infty$ then $\mu(A_n) = +\infty \; \forall n \geq n_0$ and $\mu(\bigcup_{i=1}^{\infty} A_i) = +\infty$.

2) Let now $\mu(A_i) < \infty, \forall i \geq 1$.

Then
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu(A_1 \sqcup (A_2 \setminus A_1) \sqcup (A_3 \setminus A_2) \sqcup \cdots)$$
$$= \mu(A_1) + \sum_{k=2}^{\infty} \mu(A_k \setminus A_{k-1}).$$
$$= \mu(A_1) + \sum_{k=2}^{\infty} [\mu(A_k) - \mu(A_{k-1})].$$
$$= \mu(A_1) + \lim_{n\to\infty} \underbrace{\sum_{k=2}^{n} [\mu(A_k) - \mu(A_{k-1})]}_{\mu(A_n) - \mu(A_1)}$$
$$= \lim_{n\to\infty} \mu(A_n)$$ ∎

Def $\{x_n\}_n$ is a sequence of numbers, We define

$$\liminf_n x_n := \sup_n \left(\inf_{k \geq n} x_k\right) = \sup_n \min\{x_n, x_{n+1}, \dots\}$$

$$\limsup_n x_n := \inf_n \left(\sup_{k \geq n} x_k\right).$$

Ex

① Prove that: for $x, y \in \mathbb{R}$,

$$\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$$

$$\min\{x, y\} = \frac{1}{2}(x + y - |x - y|)$$

② If f is measurable.

Prove that $f^+ = \max\{f, 0\}$ and $f^- = \min\{f, 0\}$ are measurable.

Hint: $f^+ = \frac{1}{2}(|f| + f)$; $|f| = f^+ + f^-$

$f^- = \frac{1}{2}(|f| - f)$; $f = f^+ - f^-$