3. Measures

3.1: Generalities on Measures:

Def: Let $(X, \mathcal{A})$ be a measurable space. A measure on X is a function $\mu: \mathcal{A} \to [0, \infty]$ such that:

① $\mu(\emptyset) = 0$

② For any disjoint sequence $(A_j)_j \in \mathcal{A}$

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

(σ- additive)

The set $(X, \mathcal{A}, \mu)$ will be called a measure space.

exples:

① Let X be any non empty set and let $\mathcal{A} = \mathcal{P}(X)$. For $A \in \mathcal{A}$ we define $\mu(A)$ the number of elements in A is finite and equal to $+\infty$ if not. μ is a measure on $\mathcal{A}$. This measure is called the Counting measure.

② $\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$. The measure δ_x is called the point mass at x or the Dirac measure on x.

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③ Let μ defined on $\mathcal{P}(\mathbb{R})$ by $\mu(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$

μ is finite additive but not countably additive since $\mathbb{N} = \bigcup_{j=0}^{\infty} \{j\}$, but $\mu(\mathbb{N}) = \infty \neq \sum_{j=1} \mu(\{j\}) = 0$. Then μ is not a measure.

Theorem Let μ be a measure on $(X, \mathcal{A})$. we have:

① μ is finitely additive: For any finite subsets $A_1, \dots, A_n \in \mathcal{A}$ of disjoints elements of $\mathcal{A}$, $\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j)$.

② μ is monotone, If $A, B \in \mathcal{A}$ with $A \subset B$ then $\mu(A) \leq \mu(B)$.

③ μ is countably subadditive: If $(A_j)_{j \in \mathbb{N}^*} \in \mathcal{A}$ and $A = \bigcup_{j=1}^{\infty} A_j$ then $\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$.

④ If $(A_j)_j$ is an increasing sequence in $\mathcal{A}$ and $A = \bigcup_{j=1}^{\infty} A_j$ then $\mu(A) = \lim_{n \to \infty} \mu(A_n)$. (Continuity)

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⑤ If $A, B \in \mathcal{A}$ and $A \subset B$ and $\mu(B) < \infty$ then
$$\mu(B \setminus A) = \mu(B) - \mu(A). \quad (\mu(A) < \infty \text{ suffices})$$

⑥ If $(A_j)_j$ is a decreasing sequence in $\mathcal{A}$ with $\mu(A_1) < \infty$
then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$ with $A = \bigcap_{j=1}^{\infty} A_j$. (continuity).

⚠ $X =]0,1]$, $A_n =]0, \frac{1}{n}]$, μ counting measure. $\mu(X) = \infty$
We have : $\mu(A_n) = \infty$; $(A_n) \downarrow$; $\lim_{n\to\infty} A_n = A = \{\emptyset\}$
$\mu(A) = 0$.

exercise: Show that μ is a measure on the measurable space $(X, \mathcal{B})$
iff
i) $\mu(\emptyset) = 0$
ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$
iii) If $(A_n)_n$ is an increasing sequence of the σ-algebra $\mathcal{B}$ then $\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \sup_n \left(\mu(A_n)\right)$.

Def: - We say that the measure μ is finite if $\mu(X) < \infty$.

- We say that the measure μ is σ-finite if there exists an increasing sequence $(A_j)_j$ of measurable subsets of finite measure and $\bigcup_{j=1}^{\infty} A_j = X$

- A probability measure is a measure on $(X, \mathcal{A})$ is a measure such that $\mu(X) = 1$. In this case the σ-algebra $\mathcal{A}$ is called the space of events. $\left(\mu(A^c) = \mu(X) - \mu(A) = 1 - \mu(A)\right)$

3.2 Properties of measures:

Let $(X, \mathcal{B})$ be a measurable space. We denote by $\mathcal{M}(X, \mathcal{B})$ or $\mathcal{M}(X)$ the set of measures on $(X, \mathcal{B})$. We have:

① The set $\mathcal{M}(X)$ is a convex cone. If $\mu_1, \mu_2 \in \mathcal{M}(X)$, and $\lambda > 0$ then $\mu_1 + \mu_2$, $\lambda \mu_1 \in \mathcal{M}(X)$.

We order the set $\mathcal{M}(X)$ by the relationship:
$$\mu_1 \leq \mu_2 \Leftrightarrow \mu_1(A) \leq \mu_2(A) \ \forall A \in \mathcal{B}.$$

② If $(\mu_n)_n$ is an ↗ increasing sequences of measures, then the mapping $\mu : \mathcal{B} \longrightarrow [0,\infty]$ defined by

$$\mu(A) = \lim_{n\to\infty} \mu_n(A) = \underline{\sup_n} \mu_n(A) \quad \text{for}$$

any $A \in \mathcal{B}$ is a <u>measure</u> on X.

<u>Proof</u>: . $\mu(\phi) = \lim_{n\to\infty} \mu_n(\phi) = 0$ ✓

. If $A, B \in \mathcal{B}$ and disjoints $A \cap B = \phi$,

$$\mu(A \cup B) = \lim_{n\to\infty} \mu_n(A\cup B) = \lim_n \mu_n(A) + \lim_n \mu_n(B)$$
$$= \mu(A) + \mu(B).$$

. Let $(A_n)_n$ be $\nearrow$ sequence of $\mathcal{B}$ and $A = \bigcup_{n=1}^{\infty} A_n$.

Continuity of measures

We have
$$\mu_j(A_n) \boxed{\leq} \mu(A_n) \leq \mu(A) \quad \forall j$$

$$\Rightarrow \quad \mu_j(A) = \lim_{n\to\infty} \mu_j(A_n) \leq \lim_{n\to\infty} \mu(A_n) \leq \mu(A)$$

and $$\mu(A) = \lim_{j\to\infty} \mu_j(A) \leq \lim_{n\to\infty} \mu(A_n) \leq \mu(A)$$

Then $$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

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4. Complete measure Spaces:

<u>Def</u>: Let $(X, \mathcal{B}, \mu)$ be a measure space. A subset of X is called **a null set or a negligible set** if A is contained in a measurable subset of measure zero. We denote by $\mathcal{N}$ the set of null sets.

<u>Example</u>: Let $(X, \mathcal{B})$ be a measurable space such that $\forall x \in X$, $\{x\} \in \mathcal{B}$. If we take $\mu = \delta_a$ with $a \in X$. Then every subset $A \subset \mathcal{B}$ such that $a \notin A$, is a <u>null set</u>.

⚠ 1) $\phi \in \mathcal{N}$

2) Any subset of a null set is a null set. If $A \subset B$ and $B \in \mathcal{N}$ then there is an $C \in \mathcal{B}$ such that $\mu(C) = 0$ and $B \subset C$.

3) A countable union of null sets is a null set: If $(A_n)_n$ is any sequence in $\mathcal{N}$, for each n, choose an $B_n \in \mathcal{B} / A_n \subset B_n$ and $\mu(B_n) = 0$. Now we put $B = \bigcup_n B_n \in \mathcal{B}$ and $\bigcup_n A_n \subset \bigcup_n B_n$

and $$\underbrace{\mu(\bigcup_n B_n)}_{=0} \leq \sum_n \underbrace{\mu(B_n)}_{=0}. \text{ So } \mu(B) = 0.$$

Def: If $P(x)$ is some assertion applicable to numbers x of the set X, we say that: $P(x)$ for almost every $x \in X$ or $P(x)$ a.e x or $P(x)$ for μ- almost every x, $P(x)$ μ-a.e x.
to mean that $A = \{x \in X,\ P(x) \text{ is false}\}$ is a null set.
$\subset X$

Def: A measure space $(X, \mathcal{B}, \mu)$ is said to be **complete** if any nullset is measurable ($\mathcal{N} \subset \mathcal{B}$), we say that the measure μ is complete.

Thm Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\mathcal{N}$ be the set of the null sets of X. Let $\mathcal{B}' = \left\{ A \cup B \,/\, \begin{matrix} A \in \mathcal{B} \\ B \in \mathcal{N} \end{matrix} \right\}$.
$\mathcal{B}'$ is a σ-algebra on X and there exists a unique measure μ' which extends the measure μ on the σ-algebra $\mathcal{B}'$. The measure space $(X, \mathcal{B}', \mu')$ is complete.

$\mu' = \mu$ on $\mathcal{B}$

5- Outer Mesure

Def Let X be a nonempty set. An outer measure μ^* on X is a mapping $\mu^* : \mathcal{P}(X) \longrightarrow [0, \infty]$ which fulfills the following axioms:

i) $\mu^*(\emptyset) = 0$

ii) If $(A_n)_{n \in \mathbb{N}}$ is a sequence of subsets of X, then $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

iii) μ^* is increasing (i.e $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$)

exple Any measure on $\mathcal{P}(X)$ is an outer measure.

Def: Let X be a set and μ^* be an outer measure on X. A subset A of X is called μ^*-measurable if

$$\forall B \subset X : \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Thm (Caratheodory's Construction)
Let X be a non empty set and μ^* be an outer measure on X. Then the set $\mathcal{B}'$ of the μ^*- measurable subsets is a <u>σ- algebra</u> on X and the restriction of μ^* on $\mathcal{B}'$ denoted $\mu^*_{/\mathcal{B}'}$ is a <u>complete measure</u>

<u>Proof</u>: * $\mathcal{B}'$ is a σ- algebra on X:

- ϕ is μ^*- measurable.
- If A is μ^*- measurable then A^c is μ^*- measurable
- If A, B are μ^*- measurable then $A \cup B$ is μ^*- measurable.
- If $(A_j)_j$ is a sequence of μ^*- measurable subsets then $\bigcup_{j=1}^{\infty} A_j$ is μ^*- measurable.

* μ^* is a measure on $\mathcal{B}'$.
* μ^* is <u>complete</u>. (If A is nul set, then exits $B \in \mathcal{B}'$ such that $A \subset B$ and $\mu^*(B) = 0$).

<u>Thm</u> : Let $(X, \mathcal{B}, \mu)$ be a measure space and μ- σ- finite measure. Let μ^* the outer measure defined on $\mathcal{P}(X)$ by $\mu^*(A) = \inf \{ \sum_j \mu(A_j) ; A \subset \bigcup_j A_j \text{ and } A_j \in \mathcal{B} \}$
We denote by $\hat{\mathcal{B}}$ the complete σ- algebra and $\mathcal{B}_0$ the σ- algebra of the μ^*- measurable sets then $\hat{\mathcal{B}} = \mathcal{B}_0$.

⚠
- μ^* is an outer measure.
- Any $A \in \mathcal{B}$ is μ^*- measurable, ($A \in \mathcal{B}_0$). So $\mathcal{B} \subset \mathcal{B}_0$
- $\mu^*_{/\mathcal{B}} = \mu$.

<u>proof</u>:
- $\hat{\mathcal{B}} \subset \mathcal{B}_0$: If A is a null set then A is μ^*- measurable
- $\mathcal{B}_0 \subset \hat{\mathcal{B}}$: If $A \in \mathcal{B}_0$ $\begin{cases} \mu^*(A) < \infty \rightarrow A \in \hat{\mathcal{B}} \\ \mu^*(A) = \infty \rightarrow A \in \hat{\mathcal{B}}. \end{cases}$

5.1 Monotone class and σ- Algebra

Def: A collection of sets $\mathcal{M}$ is called a monotone class if for any monotone sequence $(A_n)_n$ of $\mathcal{M}$, $\lim_{n\to\infty} A_n \in \mathcal{M}$.

examples: ① Any σ-algebra is a monotone class.

② Any arbitrary intersection of monotone classes is a monotone class.

③ If $A \subset X$, the intersection of all monotone classes that contain A is called the monotone class generated by A and denoted by $\mathcal{M}(A)$.

Thm Let $\mathcal{A}$ be an algebra of X. We denote by $\mathcal{M}(\mathcal{A})$ the monotone class generated by $\mathcal{A}$ and by $\sigma(\mathcal{A})$ the σ-algebra generated by $\mathcal{A}$. Then $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$

Proof: . $\sigma(\mathcal{A})$ is a monotone class.

. $\mathcal{A} \subset \sigma(\mathcal{A})$ then $\sigma(\mathcal{A}) \supset \mathcal{M}(\mathcal{A})$

. Now $\sigma(\mathcal{A}) \overset{?}{\subset} \mathcal{M}(\mathcal{A})$.

We define $\tilde{S} = \left\{ T \in \mathcal{P}(X) \;/\; \left.\begin{matrix} S \cup T \\ S \setminus T \\ T \setminus S \end{matrix}\right\} \in \mathcal{M}(A) \right\}$ For $S \subset X$

$\tilde{S}$ is monotonic class. $\mathcal{A} \subset \tilde{S}$

then $\mathcal{M}(\mathcal{A}) \subset \tilde{S}$ $\forall S \in \mathcal{M}(\mathcal{A})$.

We prove also $\mathcal{M}(A)$ is an algebra

by Lemma

Let $\mathcal{M}$ be an algebra closed under increasing limit then $\mathcal{M}$ is σ-algebra.

* We end this paragraph with a property of measure:

Thm Let μ_1 and μ_2 be 2 positive measures on a measurable Space $(X, \mathcal{B})$. Assume that there exists a class $\mathcal{E}$ of measurable subsets such that:

a) $\mathcal{E}$ is closed under finite intersection and that the σ-algebra generated by $\mathcal{E}$ is equal to $\mathcal{B}$.

b) There exists an increasing sequence $(E_n)_n$ in $\mathcal{E}$ such that $\lim_{n\to\infty} E_n = X$

c) $\mu_1(C) = \mu_2(C) < \infty$ for any $C \in \mathcal{E}$.

Then $\mu_1 = \mu_2$.

Proof: Consider $\mathcal{A} = \{A \in \mathcal{B} \,/\, \mu_1(A) = \mu_2(A)\}$

- $\mathcal{E} \subset \mathcal{A}$
- $\mathcal{A}$ is a monotone class
- $\mathcal{A}$ is a σ-algebra
- $\sigma(\mathcal{E}) = \mathcal{B} \subset \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{B}$ and $\mu_1 = \mu_2$.

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6. Lebesgue Measure on $\mathbb{R}$

Thm There exists only and only one measure λ on $\mathcal{B}_{\mathbb{R}}$ satisfying:

i) λ is invariant under translation

$(\forall x \in \mathbb{R}, \forall A \in \mathcal{B}_{\mathbb{R}} \; ; \; \lambda(x+A) = \lambda(A))$

ii) $\lambda([0,1]) = 1$.

Proof: uniqueness Assume there exits 2 measures μ and ν on $\mathcal{B}_{\mathbb{R}}$ satisfying (i) and (ii)

$\lambda[0,\frac{1}{n}) \leq \frac{1}{n} \Rightarrow \lambda(\{0\}) = 0$

Then any finite set or countable set is a null set and all intervals $[a,b]$, $(a,b]$, $[b,a)$ have the same measure.

use Thm 4.4:
