2.7 Product of σ-algebras

Def Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be 2 measurable spaces. We denote by $X = X_1 \times X_2$ (cartesian product). A subset $R = A_1 \times A_2$ of $X_1 \times X_2$ is called a rectangle with $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. We denote by $\mathcal{R}$ the set of all rectangles in X. The product σ-algebra of $\mathcal{A}_1$ and $\mathcal{A}_2$ on X is the σ-algebra generated by $\mathcal{R}$ and will be denoted by $\mathcal{A}_1 \otimes \mathcal{A}_2$.

⚠ In the same way if $(X_j, \mathcal{A}_j)$, $j = 1, \dots, n$ are n measurable spaces, we define the σ-algebra $\bigotimes_{j=1}^{n} \mathcal{A}_j$ on the space $X = \prod_{j=1}^{n} X_j$.

2.8 Pull back of a σ-algebra:

Let X and X' 2 non empty sets, and let $f : X \to X'$ a mapping. Let $\mathcal{B}$ be a family of subsets of X'. We define $f^{-1}(\mathcal{B}) = \{ f^{-1}(A) ; A \in \mathcal{B} \}$

Proposition: If $\mathcal{B}$ is a σ-algebra on X', then $f^{-1}(\mathcal{B})$ is a σ-algebra on X called the pull back of $\mathcal{B}$ by f.

Proof: We have $f^{-1}(X') = X$ and $\bigcup_j f^{-1}(A_j) = f^{-1}(\bigcup_j A_j)$ and

$(f^{-1}(A))^c = f^{-1}(A^c)$. ▪

If X is a subset of X' and f is an injection of X onto X' then the pull back of a σ-algebra on X' is called the trace of this σ-algebra on X.

Prop Let X and X' be 2 non empty sets and $f : X \to X'$ a mapping. Let $\mathcal{B}$ be a family of subsets of X' and $\overline{\mathcal{B}}$ the σ-algebra generated by $\mathcal{B}$. Then $f^{-1}(\overline{\mathcal{B}})$ is the σ-algebra generated by $f^{-1}(\mathcal{B})$.

3. Measures

3.1: Generalities on Measures:

Def: Let $(X, \mathcal{A})$ be a measurable space. A measure on X is a function $\mu : \mathcal{A} \to [0, \infty]$ such that:

① $\mu(\emptyset) = 0$

② For any disjoint sequence $(A_j)_j \in \mathcal{A}$

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

(σ- additive)

The set $(X, \mathcal{A}, \mu)$ will be called a measure space.

exples:

① Let X be any non empty set and let $\mathcal{A} = \mathcal{P}(X)$. For $A \in \mathcal{A}$ we define $\mu(A)$ the number of elements in A is finite and equal to $+\infty$ if not. μ is a measure on $\mathcal{A}$. This measure is called the Counting measure.

② $\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$. The measure δ_x is called the point mass at x or the Dirac measure on x.

③ Let μ defined on $\mathcal{P}(\mathbb{R})$ by $\mu(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$

μ is finite additive but not countably additive since $\mathbb{N} = \bigcup_{j=0}^{\infty} \{j\}$, but $\mu(\mathbb{N}) = \infty \neq \sum_{j=1}^{\infty} \mu(\{j\}) = 0$. Then μ is not a measure.

Theorem Let μ be a measure on $(X, \mathcal{A})$. We have:

① μ is finitely additive: For any finite subsets $A_1, \ldots, A_n \in \mathcal{A}$ of disjoints elements of $\mathcal{A}$, $\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j)$.

② μ is monotone, If $A, B \in \mathcal{A}$ with $A \subset B$ then $\mu(A) \leq \mu(B)$.

③ μ is countably subadditive: If $(A_j)_{j \in \mathbb{N}^*} \in \mathcal{A}$ and $A = \bigcup_{j=1}^{\infty} A_j$ then $\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$.

④ If $(A_j)_j$ is an increasing sequence in $\mathcal{A}$ and $A = \bigcup_{j=1}^{\infty} A_j$ then $\mu(A) = \lim_{n \to \infty} \mu(A_n)$. (Continuity)

⑤ If $A, B \in \mathcal{A}$ and $A \subset B$ and $\mu(B) < \infty$ then

$$\mu(B \setminus A) = \mu(B) - \mu(A). \quad (\mu(A) < \infty \text{ suffices})$$

⑥ If $(A_j)_j$ is a decreasing sequence in $\mathcal{A}$ with $\mu(A_1) < \infty$ then $\mu(A) = \lim_{n \to \infty} \mu(A_n)$ with $A = \bigcap_{j=1}^{\infty} A_j$. (continuity).

⚠ $X =]0,1]$, $A_n =]0, \frac{1}{n}]$, μ counting measure.
We have : $\mu(A_n) = \infty$; $(A_n) \downarrow$; $\lim_{n\to\infty} A_n = A = \{\emptyset\}$
$\mu(A) = 0$.

exercise: Show that μ is a measure on the measurable space $(X, \mathcal{B})$ iff

i) $\mu(\emptyset) = 0$
ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$
iii) If $(A_n)_n$ is an increasing sequence of the σ-algebra $\mathcal{B}$ then $\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \sup_n (\mu(A_n))$.

Def: - We say that the measure μ is finite if $\mu(X) < \infty$.

- We say that the measure μ is σ-finite if there exits an increasing sequence $(A_j)_j$ of measurable subsets of finite measure and $\bigcup_{j=1}^{\infty} A_j = X$

- A probability measure is a measure on $(X, \mathcal{A})$ is a measure such that $\mu(X) = 1$. In this case the σ-algebra $\mathcal{A}$ is called the space of events.

3.2 Properties of measures:

Let $(X, \mathcal{B})$ be a measurable space. We denote by $\mathcal{M}(X, \mathcal{B})$ or $\mathcal{M}(X)$ the set of measures on $(X, \mathcal{B})$. We have:

① The set $\mathcal{M}(X)$ is a convex cone. If $\mu_1, \mu_2 \in \mathcal{M}(X)$, and $\lambda > 0$ then $\mu_1 + \mu_2$, $\lambda \mu_1 \in \mathcal{M}(X)$.

We order the set $\mathcal{M}(X)$ by the relationship:

$$\mu_1 \leq \mu_2 \iff \mu_1(A) \leq \mu_2(A) \ \forall A \in \mathcal{B}.$$

② If $(\mu_n)_n$ is an ↗ increasing sequences of measures, then the mapping $\mu : \mathcal{B} \longrightarrow [0, \infty]$ defined by

$$\mu(A) = \lim_{n\to\infty} \mu_n(A) = \sup_n \mu_n(A) \quad \text{for}$$

any $A \in \mathcal{B}$ is a measure on X.

Proof:
. $\mu(\phi) = \lim_{n\to\infty} \mu_n(\phi) = 0$

. If $A, B \in \mathcal{B}$ and disjoints $A \cap B = \phi$,

$$\mu(A \cup B) = \lim_{n\to\infty} \mu_n(A \cup B) = \lim_n \mu_n(A) + \lim_n \mu_n(B)$$
$$= \mu(A) + \mu(B).$$

. Let $(A_n)_n$ be $\nearrow$ sequence of $\mathcal{B}$ and $A = \bigcup_n A_n$.

Continuity of measures

We have
$$\mu_j(A_n) \le \mu(A_n) \le \mu(A) \quad \forall j$$
$$\Rightarrow \quad \mu_j(A) = \lim_{n\to\infty} \mu_j(A_n) \le \lim_{n\to\infty} \mu(A_n) \le \mu(A)$$
and
$$\mu(A) = \lim_{j\to\infty} \mu_j(A) \le \lim_{n\to\infty} \mu(A_n) \le \mu(A)$$

Then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.