Algebra & σ-algebra

* Elementary Operations on Sets:

X will denote a nonempty set.
$\mathcal{P}(X)$ the collection of subsets of X.

If $A, B \in \mathcal{P}(X)$, we put

$$A \setminus B = \{ x \in A \text{ and } x \notin B \} = A \cap B^c$$

$$A \Delta B = (A \setminus B) \cup (B \setminus A) \quad \text{called}$$

Symmetric difference of B from A and if $A = X$, $X \setminus B = B^c$.

We can easily show:

$$A \setminus B = A \setminus (A \cap B) = (A \cup B) \setminus B$$

$$(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cup D)$$

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

$$(A \cap B) \Delta (A \cap C) = A \cap (B \Delta C)$$

<u>Def</u>: Characteristic function of sets:

For any subset $A \in \mathcal{P}(X)$; we denote χ_A the Characteristic function (or the indicator function) of A defined by

$$\chi_A(x) = \begin{cases} 1, & \forall x \in A \\ 0; & \forall x \notin A \end{cases}.$$

Properties:

① $A \subset B \iff \chi_A \leq \chi_B$

② $C = A \cap B \iff \chi_{A\cap B} = \chi_A \cdot \chi_B$

③ $B = A^c \iff \chi_{A^c} = 1 - \chi_A$

④ $C = A \cup B \iff \chi_C = \chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$

⑤ $C = A \setminus B \iff \chi_C = \chi_{A\setminus B} = \chi_A \cdot (1 - \chi_B)$

⑥ $C = A \Delta B \iff \chi_C = \chi_{A\Delta B} = |\chi_A - \chi_B|$

⑦ If $(A_n)_n$ is a sequence of subsets of X, then

$$\chi_{\bigcap_n A_n} = \chi_{A_1 \cap A_2 \cap \dots} = \inf_n \chi_{\{\bigcap_{p\leq n} A_p\}}$$
$$= \lim_{n\to\infty} \prod_{k=1}^{n} \chi_{A_k}$$

$$\chi_{\bigcup_n A_n} = \chi_{A_1 \cup A_2 \cup \dots} = \sup_n \{\chi_{\bigcup_{p\leq n} A_p}\}$$
$$= \lim_{n\to\infty} \chi_{\{\bigcup_{p\leq n} A_p\}}$$

⑧ If $(A_n)_n$ and $(B_n)_n$ are 2 sequences of subsets of X, then

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \Delta \left(\bigcup_{n=1}^{\infty} B_n\right) \subset \bigcup_{n=1}^{\infty} (A_n \Delta B_n).$$

Def: A family of subsets of X indexed by the set of Indexes I, is a mapping $j \longrightarrow X(j)$ from I in $\mathcal{P}(X)$. We denote $X(j) = X_j$ and the family is denoted by $(X_j)_{j\in I}$.

① The family $(X_j)_{j\in I}$ is called *finite* (resp countable) if I is finite (resp countable).

② A family $(X_j)_j$ is called <u>pairwise disjoint</u> (or simply disjoints) if $X_j \cap X_k = \phi, \forall j \neq k$.

Def:

1) Let $(f_n)_n$ be a sequence of real functions on X.

We define $\limsup_{n\to\infty} f_n = \overline{\lim_{n\to\infty}} f_n = \inf_n \sup\{f_m, m \geq n\}$

and $\liminf_{n\to\infty} f_n = \underline{\lim_{n\to\infty}} f_n = \sup_n \inf\{f_m, m\geq n\}$

These 2 limits are always exist and can take the values $\pm\infty$.

2) Let $(A_n)_n$ be a sequence of subsets of X. We define

$$\overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{\infty} A_m\right) \text{ (limit superior)}; \quad \underline{\lim_{n\to\infty}} A_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} A_m\right) \text{ (limit inferior)}$$

⚠ - If $(A_n)_n$ is a sequence of subsets of X, the set $A^* = \overline{\lim_{n\to\infty}} A_n$ is of all those points of X which belong to A_n for <u>infinitely</u> many values of n.

- The set $A_* = \underline{\lim_{n\to\infty}} A_n$ is of all those points of X which belong to A_n for all but a <u>finite</u> number of values of n.

Remarks:

① If the sequence $(f_n)_n$ converges to the function f then $\overline{\lim} f_n = \underline{\lim} f_n = f$

② $A^* = \overline{\lim} A_n = \{x \in X \,/\, \sum_{n=1}^{\infty} \chi_{A_n}(x) = \infty\}$

is the set of the elements of X which are in an <u>infinite</u> sets of A_n.

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③ $A_* = \underline{\lim} A_n = \{x \in X \,/\, \sum_{n=1}^{\infty} \chi_{A_n^c}(x) < \infty\}$

is the set of elements of X which are in all the A_n except a finite number.

④ $A_* = \underline{\lim} A_n \subset A^* = \overline{\lim} A_n$.

⑤ $\chi_{\overline{\lim} A_n} = \overline{\lim}\, \chi_{A_n}$.

⑥ $\chi_{\underline{\lim} A_n} = \underline{\lim}\, \chi_{A_n}$.

⑦ $(A_*)^c = (\underline{\lim} A_n)^c = \overline{\lim} (A_n)^c$

$(A^*)^c = (\overline{\lim} A_n)^c = \underline{\lim} (A_n)^c$

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example: $X = \mathbb{R}$, let a sequence $(A_n)_n$ of subsets of $\mathbb{R}$ be defined by $\begin{cases} A_{2n+1} = [0, \frac{1}{2n+1}] \\ A_{2n} = [0, 2n] \end{cases}$.

Then $A_* = \underline{\lim} A_n = \bigcup_{n=1}^{\infty} (\bigcap_{m=n}^{\infty} A_m)$

$= \{x \in \mathbb{R},\ x \in A_n$ for all but finitely many $n \in \mathbb{N}\}$

$= \{0\}$.

and $A^* = \overline{\lim} A_n = \bigcap_{n=1}^{\infty} (\bigcup_{m \geq n} A_m) = \{x \in \mathbb{R} \,/\, x \in A_n$ for infinitely many $n \in \mathbb{N}\}$

$= [0, \infty)$.

⚠ $(B_n)_n = (\bigcap_{m \geq n} A_m)_n$ is an increasing sequence of subsets of X.

$(C_n)_n = (\bigcup_{m \geq n} A_m)_n$ is an decreasing sequence of subsets of X.

2.2 General Properties of σ-algebra

Def: Let $\mathcal{A}$ be a collection of subsets of X. $\mathcal{A}$ is called an algebra or a field if:

① $X \in \mathcal{A}$

② if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$. (closure under complement)

③ if $A_1, \dots, A_n \in \mathcal{A}$ then $\bigcap_{j=1}^{n} A_j \in \mathcal{A}$.

(Closure under finite intersection)

$\mathcal{A}$ is called a σ-algebra or a σ-field if in addition:

④ If $(A_j)_{j \in \mathbb{N}}$ are in $\mathcal{A}$ then $\bigcap_{j=1}^{\infty} A_j \in \mathcal{A}$.

(Closure under countable intersection)

If $\mathcal{A}$ is a σ-algebra, the pair $(X, \mathcal{A})$ is called a measurable space, and the subsets in $\mathcal{A}$ are called the measurable sets.

⚠ By Complementarity:

① If $\mathcal{A}$ is an algebra then $\phi = X^c \in \mathcal{A}$

② If $\mathcal{A}$ is an algebra and $A_1, \dots, A_n \in \mathcal{A}$ then $\bigcup_{j=1}^{n} A_j \in \mathcal{A}$ (Closure under finite Union)

③ If $\mathcal{A}$ is a σ-algebra and $(A_j)_j$ is a sequence in $\mathcal{A}$, $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

Examples:

① $\mathcal{A} = \{\phi, X\}$ is an alegra and a σ-algebra. This is the smallest σ-algebra in $\mathcal{P}(X)$.

② $\mathcal{A} = \mathcal{P}(X)$ is an algebra and a σ-algebra. This is the largest σ-algebra in $\mathcal{P}(X)$.

③ Let $\mathcal{F} = \{A, B, C\}$ be a partition of X. The set

$$\mathcal{A} = \{\phi; X; A; B; C; A\cup B = C^c; A\cup C = B^c; B\cup C = A^c\}$$

is a σ-algebra.

④ - Let $X = \mathbb{R}$ and $\mathcal{A}$ the collection of subsets of X such that either A or A^c is countable or ϕ.

$\mathcal{A}$ is a σ-algebra. In fact let (A_j) be a sequence of elements of $\mathcal{A}$.

- If $\exists p \,/\, A_p$ is countable then $\bigcap_{j=1}^{\infty} A_j \subset A_p$ is countable and $\bigcap_{j=1}^{\infty} A_j \in \mathcal{A}$.
- If $\forall A_j$ is not countable, then all A_j^c are countable and then $\bigcup_{j=1}^{\infty} A_j^c$ is countable subset of $\mathbb{R}$ and then $\bigcap_{j=1}^{\infty} A_j \in \mathcal{A}$.

- Let X be an infinite set and let $\mathcal{A}$ the collection of subsets A of X such that either A or A^c is finite, then $\mathcal{A}$ is an algebra but it is not a σ-algebra.

2.4 σ-Algebra Generated by a Subset $P \subset \mathcal{P}(X)$:

Def: Let X be a non empty set and $\mathcal{A}_1$, $\mathcal{A}_2$ 2 σ-algebras on X. We say that $\mathcal{A}_1$ is finer the $\mathcal{A}_2$ if any element of $\mathcal{A}_1$ is an element of $\mathcal{A}_2$. In this case we write $\mathcal{A}_1 \subset \mathcal{A}_2$.

⚠ Any intersection of algebras (resp σ-algebras) is an algebra (resp σ-algebra).

Def: Let X be a non empty set and $\mathcal{B} \subset \mathcal{P}(X)$. There exists a smallest algebra (resp σ-algebra) denoted by $A(\mathcal{B})$, (resp $\sigma(\mathcal{B})$) that contains $\mathcal{B}$. This algebra (resp σ-algebra) is called the algebra (resp σ-algebra) generated by $\mathcal{B}$. $A(\mathcal{B})$ (resp $\sigma(\mathcal{B})$) is the intersection of all the algebras on X (resp σ-algebra) containing $\mathcal{B}$. So this is the smallest algebra which contains $\mathcal{B}$.

examples ① Let A be a subet of X with $A \neq \emptyset$ and $A \neq X$. The σ-algebra generated by $\{A\}$ is

$\sigma(\langle A \rangle) = \{\Phi; X; A; A^c\}$.

② Let X be a nonempty set and $(P_j)_{j \in J}$ is a finite Partition of X. The algebra generated by (P_j) is constituted by the subsets of the form $\bigcup_{j \in I} P_j$ where $I \subset J$.

We remark $\mathcal{P}(J) \longrightarrow \mathcal{P}(X)$ is an isomorphism

$I \longrightarrow \bigcup_{j \in I} P_j$

If J contains n elements $(|J| = n)$ then the algebra contains 2^n elements.

2.5 Borelian σ-Algebra in $\mathbb{R}$:

If $X = \mathbb{R}$ and $\mathcal{B}$ is the σ-algebra generated by the family $\{[a,b) \ ; \ a, b \in \mathbb{R}\}$. This σ-algebra is denoted by $\mathcal{B}_{\mathbb{R}}$ and called the σ-algebra of Borel subsets on $\mathbb{R}$. ($\mathcal{B}_{\mathbb{R}}$ contains all open and closed subsets of $\mathbb{R}$). Every element

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of $\mathcal{B}_{\mathbb{R}}$ is called a Borel subset of $\mathbb{R}$.

$[a,b)$

$\bigcap_{n=1}^{\infty} [a, b+\frac{1}{n}) = [a,b]$

We can prove that:

$\mathcal{B}_{\mathbb{R}}$ is generated by $\{[a,b], \ a, b \in \mathbb{R}\}$

" " " open subsets in $\mathbb{R}$.

" " " closed subsets in $\mathbb{R}$

" " " $\{(a, \infty), \ a \in \mathbb{R}\}$

" " " $\{(-\infty, a], \ a \in \mathbb{R}\}$.

2.6 Borelian σ-Algebra in a Topological Space:

Let X be a topological space and $\mathcal{A}$ be the family of the open subsets of X. Let $\mathcal{B}$ be the σ-algebra generated by the family $\mathcal{A}$. Then $\mathcal{B}$ is called the σ-algebra of Borel subsets on X and denoted by $\mathcal{B}_X$. All open and Closed subsets of X are Borel subsets.

The family of the closed subsets of X generates $\mathcal{B}_X$.
